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THEOREMS OF THE GENERALIZED TAUBERIAN TYPE FOR MEASURES

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ABSTRACT

A simple proof of Tauberian type theorems for measures is given. The used limit is general enough to allows the approach not only to the vertex of the cone, but also to any point of the boundary of the conjugate cone.

1. INTRODUCTION

V.S. Vladimorov [7] proved theorems of the Abelian and Tauberian type for positive measures starting from their applications especially in the quantum field theory and also in order to solve some convolution equations. Vladimirov's paper opened up much research in this direction. We shall mention only the results of Yu.N.Drožinov and B. Zavjalov [2], [3]. They proved theorems of the Abelian and Tauberian type for tempered distributions and then, as a special case, they applied these results to measures improving those of Vladimirov.

Our paper [6] relates also to Vladimirov's results. We shall choose another way: we shall prove first some theo-

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rems for measures and then we shall enlarge them to some classes of generalized functions. In such a way we can start with minimum suppositions and with a simpler proof adding new suppositions in relation with the larger class of distributions. In this paper we shall enlarge the limit process in such a way that we can use it to analyse what happens when we approach not only the vertex of cone Γ^* but any point of its boundary.

2. NOTATIONS

Let Γ be a closed and acute cone in R^n with vertex in zero. $\Gamma^* = \{y \in R^n,\ (y,x) \geq 0,\ x \in \Gamma\}$ be the conjugate cone of the cone Γ. We know that for an acute cone Γ $\operatorname{int}\Gamma^* \neq \emptyset$ and we denote it by C; Γ^* is closed and convex; let $\operatorname{pr} C =$ $= \{e \in C, |e| = 1\}$.

H_e^+ be the half space $\{t \in R^n,\ (e,t) \geq 0\}$; if $e \in \Gamma^*$, then $\Gamma \subset H_e^+$.

$$J_k = \{1,2,\dots,k\};\quad I^n = \{t \in R^n,\ 0 \leq t_i \leq 1,\ i \in J_n\}\ ;$$

$$I^n(u,v) = \{t \in R^n,\ 0 \leq u_i < t_i < v_i \leq 1,\ i \in J_n\}\ ;$$

$$D_{m,k} = \{x \in I^m,\ x_{k+1} = \dots = x_m = 1\},$$

$$D_{m,k}(u,v) = \{x \in I^m(u,v), x_{k+1} = \dots = x_m = 1\}\ .$$

ρ_γ be a regular varying function of the power γ:

$$\lim_{t \to 0^+(\infty)} \rho(ut)/\rho(t) = u^\gamma\ ,\ u > 0.$$

3. THEOREM ON ASYMPTOTIC BEHAVIOUR OF THE LAPLACE TRANSFORM OF A MEASURE

THEOREM 1. *Let us suppose:*

– $\{\sigma_i\}_{i=1}^n$ *be linear independent elements from the convex closed cone* Γ^* ;

– $\rho(r) = \rho_1(r_1)\dots\rho_m(r_m)$; $\rho_i(r_i)$ *be regular varying functions of powers* $\alpha_1,\dots,\alpha_k > 0$; $\alpha_{k+1}=\dots=\alpha_m=0$, $m \leq n$;

- $g(x) \geq 0$ *be a bounded semicontinuous function over* I^n, *continuous for almost all* $x \in I^n$ *and that the point* $(1,\ldots,1)$ *is not an accumulation point of discontinuities;*
- μ *be a nonnegative measure with support in* Γ, $\mu \not\equiv 0$;
- $\tilde{\mu}(iy) = \int_\Gamma e^{-(y,t)} d\mu(t)$ *exists for all* $y \in C = \operatorname{int}\Gamma^*$.

If for $y_r = \sum_{i=1}^{m} r_i \mu_i \sigma_i + \sum_{i=m+1}^{n} \mu_i \sigma_i \equiv y_r^m + \xi$, $\mu_i > 0$, $i \in J_m$; $\mu_i \geq 0$, $i \in J_n$, there exists

$$(1) \qquad \lim_{r \to 0^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r,t)} d\mu(t) = h(y),$$

then

$$(2) \qquad \lim_{r \to 0^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r,t)} g(Z_1,\ldots,Z_m) d\mu(t) =$$

$$= \begin{cases} \dfrac{1}{\prod^k (\alpha_i)} h(y) \displaystyle\int_{(R^+)^k} \tau_1 \cdots \tau_k \, g(\tau_1,\ldots,\tau_m) \prod^k t_i^{\alpha_i - 1} dt, & k \geq 1, \\ g(1,\ldots,1) h(y), & k = 0; \end{cases}$$

where $\tau_i = e^{-t_i}$, $i \in J_k$; $\tau_i = 1$, $i \in J_m \setminus J_k$ and $Z_i = e^{-r_i \mu_i (\sigma_i, t)}$.

We will give the proof of this theorem by using three lemmas as follows. The first one is a generalization of a lemma proved by J.Karamata [5].

LEMMA 1. *If* $g(x)$ *is defined over* I^m, *continuous for almost all* $x \in D_{m,k}$ *and bounded over* $D_{m,k}$, *then for* $\alpha_i > 0$, $i \in J_k$, $1 \leq k \leq m$, *and* $\varepsilon > 0$ *there exist polynomials* $p(x_1,\ldots,x_m)$ *and* $P(x_1,\ldots,x_m)$ *such that*

$$(3) \qquad p(x_1,\ldots,x_m) \leq g(x) \leq P(x_1,\ldots,x_m)$$

and

$$(4)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1} \left[P(\tau_1,\ldots,\tau_m) - p(\tau_1,\ldots,\tau_m)\right]dt < \varepsilon$$

where

$$\tau_i = e^{-t_i},\ i\in J_k \quad\text{and}\quad \tau_i = 1,\quad i\in J_m\setminus J_k.$$

P r o o f. We shall divide our proof into three parts like J.Karamata. First we suppose that $g(x)$ is of the form:

$$(5)\qquad g(x) = \begin{cases} 1, & x\in D_{m,k}(u,v) \\ 0, & x\in I^m\setminus D_{m,k}(u,v) . \end{cases}$$

For every $\omega > 0$ there exist nonnegative numbers ε', ε'' and a continuous function $h(x)$, $0 \le h(x) \le 1$, $x\in I^m$; $g(x) = h(x)$, $x\in D_{m,k}(u,v)$ and $h(x) = 0$, $x\in I^m\setminus D_{m,k}(u-\varepsilon', v+\varepsilon'')$ and such that

$$(6)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\left[h(\tau_1,\ldots,\tau_m) - g(\tau_1,\ldots,\tau_m)\right]dt < \omega .$$

The Stone Weierstrass theorem says that there exists a polynomial $Q_\varepsilon(x_1,\ldots,x_m)$ such that

$$|Q_\varepsilon(x_1,\ldots,x_m) - h(x)| < \varepsilon ,\quad x\in I^m.$$

We can take now $P(x_1,\ldots,x_m) = Q_\varepsilon(x_1,\ldots,x_m) + \varepsilon$ and in this case $g(x) \le h(x) \le P(x_1,\ldots,x_m)$, $x\in I^m$, and

$$\int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\left[P(\tau_1,\ldots,\tau_m) - g(\tau)\right]dt \le$$

$$\le \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1} |Q_\varepsilon(\tau_1,\ldots,\tau_m) - h(\tau)|\,dt +$$

$$(7)\qquad + \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\left[h(\tau) - g(\tau)\right]dt +$$

$$+\ \varepsilon \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1} dt \le \omega + 2\varepsilon \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1} dt,$$

$$\tau = (e^{-t_1},\ldots, e^{-t_k}, 1,\ldots,1) .$$

In the same way we can find a polinomial $p(x_1,\dots,x_m)$ such that $p(x_1,\dots,x_m) \le g(x)$, $x \in I^n$ and

$$(8) \qquad \int_{(R^+)^k} \tau_1 \dots \tau_k \prod t_i^{\alpha_i - 1} [g(\tau) - p(\tau_1,\dots,\tau_m)] dt \le \omega + 2\varepsilon \prod^k \Gamma(\alpha_i).$$

The same can be proved for a step function with a finite number of jumps in every coordinate.

There only remains to suppose that our function $g(x)$ has the properties fixed in the lemma.

We can define δ and ρ in such a way that

$$(9) \qquad 2M\Big[\int_{\delta I^k} + \int_{(R^+)^k \setminus \rho I^k} \tau_1 \dots \tau_k \prod^k t_i^{\alpha_i - 1} dt \Big] < \varepsilon/6$$

where $M = \sup|g(x)|$, $x \in I^m$.

Over the bounded set $I^m(\delta,\rho)$ the function

$$(10) \qquad \tau_1 \dots \tau_k g(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i - 1}$$

has the Riemann integral; so we have ([4] p.69) two step functions with a finite number of jumps, $g_1(x)$ and $g_2(x)$, such that $g_1(x) \le g(x) \le g_2(x)$, $x \in D_{m,k}(\delta,\rho)$ and

$$(11) \qquad \int_{I^k(\delta,\rho)} \tau_1,\dots,\tau_k \prod^k t_i^{\alpha_i - 1} [g_2(\tau_1,\dots,\tau_m) - $$
$$- g_1(\tau_1,\dots,\tau_k)] dt < \varepsilon/6 .$$

The functions g_1 and g_2 can be extended over $(R^+)^k \setminus I^k(\delta,\rho)$ by the constant M. The first part of this proof says that there exist polynomials $P(x_1,\dots,x_m)$ and $p(x_1,\dots,x_m)$ such that

$$p(x_1,\dots,x_m) \le g_1(x) \le g(x) \le g_2(x) \le P(x_1,\dots,x_m)$$

and

$$(12) \qquad \int_{(R^+)^k} \tau_1 \dots \tau_k \prod^k t_i^{\alpha_i - 1} [P(\tau_1,\dots,\tau_m) - g_2(\tau)] dt < \varepsilon/3$$

$$(13)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^k t_i^{\alpha_i-1}\,[g_1(\tau)-p(\tau_1,\ldots,\tau_m)]\,dt < \varepsilon/3 .$$

From relations (9),(10),(11),(12) and (13) there follows relation (4) which had to be proved.

LEMMA 2. *Let* $g(x)$ *be a bounded, semicontinuous function over* I^m, *continuous for almost all* $x\in I^m$ *and the point* $(1,\ldots,1)$ *not an accumulation point of discontinuities. Then for* $\varepsilon>0$ *there exist polynomials* $p(x_1,\ldots,x_m)$ *and* $P(x_1,\ldots,x_m)$ *such that* $p(x_1,\ldots,x_m)\le g(x)\le P(x_1,\ldots,x_m)$ *for* $x\in I^m$ *and* $P(1,\ldots,1)-p(1,\ldots,1)<\varepsilon$.

P r o o f. From our suppositions it follows that there exist an interval $I^n(1-\omega,1)$ and a continuous function $h(x)$, $x\in I^m$ such that $g(x)\le h(x)$, $x\in I^m$, $g(x)=h(x)$, $x\in I^m(1-\omega,1)$. By the Stone-Weierstrass theorem there exist two polynomials $p(x_1,\ldots x_m)$ and $P(x_1,\ldots,x_m)$ such that

$$0\le p(x_1,\ldots,x_m)-h(x)\le\varepsilon/2,\qquad x\in I^m,$$
$$0\le h(x)-p(x_1,\ldots,x_m)<\varepsilon/2,\qquad x\in I^m .$$

These two polynomials satisfy the conditions of our lemma.

LEMMA 3. *Let us suppose that* $\{\sigma_i\}_{i=1}^n$ *are linear independent elements from* Γ^*; $g(x)\ge 0$ *is a bounded and upper (lower) semicontinuous function for* $x\in I^m$.

If for all $y\in C$, $y=\sum_{i=1}^n \mu_i\sigma_i$, $\mu_i\ge 0$ *there exists the integral*

$$(14)\qquad \int_\Gamma e^{-(y,t)}\,d\mu(t)$$

then the integral

$$(15)\qquad \int_\Gamma e^{-(y,t)}g(z_1,\ldots,z_m)\,d\mu(t)$$

exists too, where $z_i=e^{-\mu_i(\sigma_i,t)}$, $i=1,\ldots,m$.

P r o o f. By supposition on $g(x)$, $g(e^{-\mu_1(\sigma_1,t)},\ldots$ $\ldots,e^{-\mu_m(\sigma_m,t)})$, for a fixed $y \in C$, is upper (lower) semicontinuous in $t \in \Gamma$ and therefore ([4] p. 96) μ-measurable on every closed and bounded subset E of Γ and the integral

$$\int_E e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t)$$

exists for every such E ([4], p. 112). Let us denote by $y_p^m = \sum_{i=1}^{m} p_i\mu_i\sigma_i$. We know that $y_p^m \in C$ for all $p \geq 0$; for $\xi \in C$, $y_p^m + \xi$ belongs to C too, because Γ^* is convex. The integral (14) exists for all $y \in C$ and

$$\int_\Gamma e^{-(y_{p+1}^m+\xi,t)}\,d\mu(t) = \int_\Gamma e^{-(y,t)}\,e^{-(y_p^m,t)}\,d\mu(t) \ .$$

From this relation it follows that for every polynomial $P(x_1,\ldots,x_m)$, $m \leq n$ there exists the following integral too:

$$\int_\Gamma e^{-(y,t)} p(z_1,\ldots,z_m)\,d\mu(t)$$

where $z_i = e^{-\mu_i(\sigma_i,t)}$.

Let $P(x_1,\ldots,x_m)$ be the polynomial from Lemma 1. then

$$\int_E e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t) \leq \int_\Gamma e^{-(y,t)} P(z_1,\ldots,z_m)\,d\mu$$

for every $E \subseteq \Gamma$ which shows that the integral

$$\int_\Gamma e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t)$$

exists.

P r o o f of Theorem 1. Case $k \geq 1$. Let us suppose that $P(x_1,\ldots,x_m)$ and $p(x_1,\ldots,x_m)$ are polynomials from Lemma 1, then by our suppositions and Lemma 3 we have:

$$\begin{aligned}(16)\quad &\int_\Gamma e^{-(y_p^m+\xi,t)} p(z_1,\ldots,z_m)\,d\mu(t) \leq \int_\Gamma e^{-(y_p^m+\xi,t)} g(z_1,\ldots,z_m)\,d\mu(t)\\ &\leq \int_\Gamma e^{-(y_p^m+\xi,t)} P(z_1,\ldots,z_m)\,d\mu(t) \ ,\end{aligned}$$

where $z_i = e^{-p_i\mu_i(\sigma_i,t)}$; similarly:

$$\int_{(R^+)^k} \tau_1 \dots \tau_k \prod^k t_i^{\alpha_i-1} p(\tau_1,\dots,\tau_m)dt \le$$

$$(17) \qquad \le \int_{(R^+)^k} \tau_1 \dots \tau_k \prod^k t_i^{\alpha_i-1} g(\tau)dt \le$$

$$\le \int_{(R^+)^k} \tau_1 \dots \tau_k \prod^k t_i^{\alpha_i-1} P(\tau_1,\dots,\tau_m)dt$$

where $\tau = (e^{-t_1},\dots,e^{-t_k},1,\dots,1)$.

Now, in relation (2) instead of r_i, $i \in J_m$, we write $(n_i+1)r_i$, $i \in J_m$, then

$$\lim_{r\to o^+ (\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} e^{-(y_{nr}^m,t)} d\mu(t) =$$

$$(18) \qquad = h(y) = \frac{1}{\prod^k (n_i+1)^{\alpha_i}} .$$

This relation shows that for any polynomial $P(x_1,\dots,x_m)$ we have

$$\lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} P(Z_1,\dots,Z_m)d\mu(t) =$$

$$(19) \qquad = \frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1 \dots \tau_k P(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt .$$

From relations 16-19 it follows:

$$\frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1 \dots \tau_k p(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt \le$$

$$(20) \quad \le \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} g(Z_1,\dots,Z_m)d\mu(t) \le$$

$$\frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1 \dots \tau_k P(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt . .$$

Now, it is enough to use the properties of our polynomials p and P from Lemma 1, and we shall have relation (2) that we had to prove.

The difference of the proof in the case when no $\alpha_i \neq 0$ is only in the fact that we shall use Lemma 2 instead of Lemma 1.

Relation (18) in this case becomes

$$(21) \qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} e^{-(y_{nr}^m,t)} d\mu(t) = h(y) .$$

For any polynomial $P(x_1,\ldots,x_m)$ we have

$$(22) \qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} P(Z_1,\ldots,Z_m) d\mu(t) =$$

$$= h(y)P(1,\ldots,1) ,$$

where $Z_i = e^{-r_i\mu_i(\sigma_i,t)}$, $i \in J_m$.

Now the proof follows as in the first case.

4. THEOREMS OF THE TAUBERIAN TYPE FOR THE LAPLACE TRANSFORM OF A MEASURE

THEOREM 2. *Let us suppose:*

- $\{\sigma_i\}_{i=1}^n$ *are linear independent elements from the convex closed cone* Γ^* *;*
- $\rho(r) = \rho_1(r_1)\ldots\rho_m(r_m)$; $\rho_i(r_i)$ *are regular varying functions of powers* $\alpha_1,\ldots,\alpha_k > 0$; $\alpha_{k+1}=\ldots=\alpha_m=0$, *respectively.*
- μ *is a nonnegative measure with a support in* Γ, $\mu \neq 0$;
- $\tilde{\mu}(iy)$ *exists for all* $y \in C$.

If there exists for a fixed $y \in C$, $y = \sum_{i=1}^n \mu_i\sigma_i$, $\mu_i \geq 0$

$$(23) \qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu(t) = h(y) ,$$

where $y_r^m = \sum_{i=1}^n r_i\mu_i\sigma_i$ *and* $\xi = \sum_{i=m+1}^n \mu_i\sigma_i$, *then*

$$(24)\qquad \lim_{r\to\infty(o^+)} \rho(1/r) \int\limits_{\Gamma\cap\left(\bigcap\limits^{m} r_i \frac{\sigma_i}{\mu_i|\sigma_i|^2} - H^+_{\sigma_i}\right)} e^{-(\xi,t)}\,d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod\limits^{k}\Gamma(\alpha_i+1)}, & k \geq 1 \\ h(y), & k = 0\,. \end{cases}$$

To prove this theorem we shall use our Theorem 1 and the lemma as follows.

LEMMA 4. *Let* $g(y)$ *be the function defined on* I^m *:*

$$g(y) = \begin{cases} \prod\limits^{m} y_i^{-1}, & e^{-1} \leq y_i \leq 1, \text{ for all } i \in J_m \\ 0, & 0 \leq y_i < e^{-1}, \text{ for one } i \in J_m \end{cases}$$

then

$$z_1 \ldots z_m\, g(z_1,\ldots,z_m) = \emptyset_{\Gamma\cap\left[\bigcap\limits^{m}\left(\frac{1}{q_i} e_i - H^+_{e_i}\right)\right]}$$

for $e_i \in \operatorname{pr}\Gamma^*$, $i \in J_m$ *and* $t \in \Gamma$, *where* $\emptyset_F$ *is the characteristic function of the set* F *and*

$$z_i = e^{-(e_i, q_i t)}, \quad q_i > 0\,.$$

P r o o f. By our supposition on g we have:

$$z_1 \ldots z_m g(z_1,\ldots,z_m) = \begin{cases} 1, \text{ if } 0 \leq (e_i, tq_i) \leq 1, \text{ for all } i \in J_m \\ 0, \text{ if } 1 < (e_i, tq_i), \text{ for one } i \in J_m\,. \end{cases}$$

The inequality $0 \leq (e_i, tq_i) \leq 1$ is equivalent with

$$0 \leq \left(\frac{1}{q_i} e_i, t\right) \leq \left(\frac{1}{q_i}\right)^2\,.$$

The first part is always satisfied for $t \in \Gamma$. For the second part we have

$$\left(\frac{1}{q_i} e_i, \frac{1}{q_i} e_i - t\right) \geq 0\,,$$

whence

$$\frac{1}{q_i} e_i - t \in H^+_{e_i} \quad \text{or} \quad t \in \left(\frac{1}{q_i} e_i - H^+_{e_i} \right).$$

It follows that t belongs to Γ and to every halfspace $\frac{1}{q_i} e_i - H^+_{e_i}$.

P r o o f of Theorem 2. If we take in Theorem 1 the function g as in our Lemma 4, then from relation (2) with $r_i = 1/r_i$ we have:

$$\lim_{r\to\infty(o^+)} \rho(1/r) \int_\Gamma e^{-(\xi,t)} Z_1 \ldots Z_m g(Z_1 \ldots Z_m) d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod\limits^{k} \Gamma(\alpha_i)} \displaystyle\int_{(R^+)^k} \tau_1 \ldots \tau_k g(\tau_1, \ldots, \tau_m) \prod^{k} t_i^{\alpha_i - 1} dt \\ h(y)\ g(1, \ldots, 1)\ , \end{cases}$$

where $Z_i = e^{-(\sigma_i, r_i^{-1} \mu_i t)}$; $\tau_i = e^{-t_i}$, $i \in J_k$ and $\tau_i = 1$, $i \in J_m \setminus J_k$.

By Lemma 4 we have:

$$\lim_{r\to\infty(o^+)} \rho(1/r) \int_{\Gamma \cap \left[\bigcap\limits^{m} \left(r_i \frac{\sigma_i}{\mu_i |\sigma_i|^2} - H^+_{\sigma_i}\right)\right]} e^{-(\xi,t)} d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod\limits^{k} \Gamma(\alpha_i)} \displaystyle\int_{I^k} \prod^{k} t_i^{\alpha_i - 1} dt, & k \geq 1 \\ h(y), & k = 0. \end{cases}$$

whence follows relation (24) of our theorem.

REMARKS. The nonnegativity of the measure μ in Theorem 2 can be replaced by a less restrictive condition as follows:

Let $\rho'(r)$ be the product of regular varying functions $\rho_i'(r_i)$ of powers $\alpha_i' > 0$, $i \in J_k$. We know that every real measure μ is the difference of two nonnegative measures μ^+ and μ^-, $\mu = \mu^+ - \mu^-$.

Theorem 2 remains valid if instead of the nonnegativity of μ we suppose that μ^- is with support in Γ and that there exists ρ' such that

a) $\alpha_i' \le \alpha_i$ and

$$(25)\qquad \lim_{r\to o^+} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu^-(t) = h^-(y) \neq 0$$

or

b) $\alpha_i \le \alpha_i'$ and

$$(26)\qquad \lim_{r\to\infty} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu^-(t) = h^-(y) \neq 0 .$$

In case a) in our Theorem 2 we have use the limit only with the first value, and in case b) with the one which is in brackets.

The following function shows the interest of our Theorem 2. The two-dimensional Laplace transform of the function $\frac{1}{\sqrt{x+y}} J_1(2\sqrt{x+y})$ is ([1], p.241): $\frac{e^{-(1/u)}-e^{-(1/v)}}{u-v}$ where J_1 is the Bessel function. There is no $\alpha \geq 0$ such that

$$\lim_{\rho\to o^+} \rho^\alpha \frac{e^{-\frac{1}{\rho u}} - e^{-\frac{1}{\rho v}}}{\rho(u-v)} = h \neq 0$$

$u,v \neq 0$, $u \neq v$.

But we can use our Theorem 2. (See Remarks). We have to take that $\Gamma^* = (R^+)^2$, $\sigma_1 = (1,0)$, $\sigma_2 = (0,1)$. It is easy to see that

$$\lim_{r\to o^+} \frac{e^{-\frac{1}{ru}} - e^{-\frac{1}{v}}}{ru-v} = \frac{e^{-\frac{1}{v}}}{v}$$

and

$$\lim_{r\to o^+} \frac{e^{\frac{1}{u}} - e^{-\frac{1}{rv}}}{u-rv} = \frac{e^{-\frac{1}{u}}}{u} .$$

There exists one and only one element t_o which belongs to all hyperplanes $q_i\sigma_i - H_{\sigma_i}$, $q_i \neq 0$, $i \in J_n$, because the system

$$(q_i\sigma_i, t) = (q_i)^2, \; q_i \neq 0, \quad i \in J_n \tag{27}$$

has one and only one solution t_o. The set $\bigcap\limits^{n} (q_i\sigma_i - H^+_{\sigma_i})$ is a cone translated in the point t_o. To show this let us suppose that $t \in \bigcap\limits^{n} (q_i\sigma_i - H^+_{\sigma_i})$, then $t_o - t$ belongs to $\bigcap\limits^{n} H^+_{\sigma_i}$:

$$\begin{aligned}(q_i\sigma_i, t_o - t) &= (q_i\sigma_i, t_o - t) + (q_i\sigma_i, q_i\sigma_i - t_o)\\ &= (q_i\sigma_i, q_i\sigma_i - t) \geq 0, \quad i \in J_n .\end{aligned}$$

In the case $\Gamma = (R^+)^n$, $\sigma_i = e_i = (0,\ldots,1,\ldots,0)$ the set $\bigcap\limits^{n} q_ie_i - H^+_{e_i}$ is the cone $-(R^+)^n$ translated in the point $(q_1, \ldots, q_n)$.

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REZIME

GENERALISANE TAUBEROVE TEOREME ZA MERE

Dat je jednostavan dokaz Tauberove teoreme za meru koja je nenegativna ili zadovoljava dodatni uslov. Granični proces je opštiji i dozvoljava da se ispituje šta se dešava kada se približimo ne samo vrhu konjugovanog konusa, već i bilo kojoj tački njegovog ruba.

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SOME APPLICATIONS OF A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT

In [3] a fixed point theorem for multivalued mappings in not necessarily locally convex topological vector spaces is proved. Here we obtain, by using this fixed point theorem and the dual result, two theorems on the coincidence point, a result on the equilibrium state in a special noncooperative game and three existence theorems for some classes of equations.

1. NOTATIONS AND DEFINITIONS

Recently some fixed point theorems for multivalued mappings in not necessarily locally convex topological vector spaces have been proved ([3], [4], [5], [8], [9], [10], [11], [13]). Some applications in the theory of optimization are given in [6] and [12]. This paper contains some further applications of fixed point theorems from [3] and [11]. The following notations and definitions are taken from [15] and [6]. In this paper it will be assumed that all topological vector spaces are Hausdorff. If X is a topological space, by 2^X we shall denote the family of all nonempty subsets of X and by 2^X_c the family of

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all nonempty closed subsets of X. The family of all nonempty, closed and convex subsets of X will be denoted by $\mathcal{K}(X)$. If X is a topological vector space and $A \subseteq X$ then co A denotes the convex hull of A.

Let $f: X \to Y$, where X and Y are topological spaces. Then for every $A \subseteq X$, $B \subseteq Y$:

$$f(A) = \bigcup_{x \in A} f(x), \quad f^{-1}(B) = \{x \mid x \in X,\ f(x) \cap B \neq \emptyset\} .$$

The mapping $f: X \to Y$ is upper semicontinuous if and only if for each closed set $B \subseteq Y$, the set $f^{-1}(B)$ is a closed subset of X.

DEFINITION. *Let* E *be a topological vector space,* $\mathcal{U}$ *be the fundamental family of neighbourhoods of zero in* E *and* $K \subseteq E$. *We say that the set* K *is of Zima's type if and only if for every* $V \in \mathcal{U}$ *there exists* $U \in \mathcal{U}$ *so that* $\mathrm{co}(U \cap (K-K)) \subseteq V$.

Some examples of subsets of Zima's type in not necessarily locally convex topological vector spaces are given in [6]. Let E be a vector space and $\| \ \|^*: E \to [0,\infty)$ so that the following conditions are satisfied:

1. $\|x\|^* = 0 \iff x = 0$.
2. $\|x\|^* = \|-x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_o\|^* \to 0$ and $\lambda_n \to \lambda_o$, when $n \to \infty$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

Then $(E, \| \ \|^*)$ is a paranormed space. This is a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\mathcal{V} = \{V_r\}_{r>0}$, where:

$$V_r = \{x \mid x \in E,\ \|x\|^* < r\} .$$

In [16] Zima proved a generalization of the Schauder fixed point theorem for the mapping $f: K \to K$, where K is a subset

of E and $(E, \| \ \|^*)$ is a paranormed space and there exists $C > 0$ so that

(1) $\| tx \|^* \leq C\, t \| x \|^*$, for every $t \in [0,1]$ and $x \in f(K)-f(K)$.

It is easy to see that the inequality (1) implies that f(K) is of Zima's type. In [6] an example of E and K is given, where $K \subseteq E$ and $(E, \| \ \|^*)$ is a non-locally convex paranormed space so that :

$$\| tx \|^* \leq C\, t \| x \|^* \text{ for every } t \in [0,1] \text{ and } x \in K-K .$$

An example of K and E is given in [9] and [16].

In [3] the following fixed point theorem is proved.

THEOREM A. *Let E be a topological space, $\mathcal{U}$ the fundamental system of neighbourhoods of zero in E, K a closed and convex subset of E, $f: K \to \mathcal{R}(K)$ an upper semicontinuous mapping such that $\overline{f(K)}$ is compact and $f(K)$ is of Zima's type. Then there exists $x \in K$ so that $x \in f(x)$.*

2. TWO THEOREMS ON THE COINCIDENCE POINT

Using the same method as in [15], we shall prove the following coincidence theorem.

THEOREM 1. *Let E be a topological vector space, S a nonempty closed and convex subset of E, K a compact subset of S, H a topological space and $\phi: S \to 2^H_C$, $\psi: K \to 2^H_C$ upper semicontinuous mappings such that $\psi^{-1}(\phi(S))$ is of Zima's type. Let for every $x \in S$:*

(i) $\phi(x) \cap \psi(K) \neq \emptyset$.

(ii) $\psi^{-1}(\phi(x))$ *be convex.*

If H is regular or H is Hausdorff and $\psi(x)$ is compact for every $x \in K$, there exists $x_0 \in K$ such that $\phi(x_0) \cap \psi(x_0) \neq \emptyset$.

P r o o f. Let us define, as in [15], the mapping $\hat{\phi}:S \to 2^K_c$ in the following way:

$$\hat{\phi}(x) = \psi^{-1}(\phi(x)), \quad x \in S .$$

It remains to be proved that $\hat{\phi}$ satisfies all the conditions of Theorem A. Since $\psi^{-1}(\phi(S))$ is of Zima´s type, we shall prove that $\hat{\phi}$ is upper semicontinuous and $\hat{\phi}(x) \in R(K)$ for every $x \in S$. Since (ii) holds, the relation $\hat{\phi}(x) \in R(K)$ follows from the upper semicontinuity of ψ and the closedness of the set $\phi(x)$. The upper semicontinuity of $\hat{\phi}$ follows as in [15], since

$$\begin{aligned}\hat{\phi}^{-1}(A) &= \{x \mid x \in S,\ \hat{\phi}(x) \cap A \neq \emptyset\} = \\ &= \{x \mid x \in S,\ \psi^{-1}(\phi(x)) \cap A \neq \emptyset\} = \\ &= \{x \mid x \in S,\ \phi(x) \cap \psi(A) \neq \emptyset\}\end{aligned}$$

where A is a closed subset of K.

COROLLARY 1. *Let* X *be a topological vector space,* L *a nonempty, closed and convex subset of* X, $f:L \to R(X)$ *an upper semicontinuous mapping such that* $\overline{f(L)}$ *is compact, and* G *a linear one to one mapping from* X *onto* X *such that* G *and* G^{-1} *are continuous and* $f(L) \subseteq G(L)$. *If* f(L) *is of Zima´s type, there exists* $x \in L$ *such that* $G(x) \in f(x)$.

P r o o f. Let $H = X, S = L, K = G^{-1}\overline{f(L)}$, $\phi = f$ and $\psi = G$. From the compactness of $\overline{f(L)}$, it follows that K is a compact subset of L. Since G^{-1} is a linear mapping and $f(x) \in R(X)$ for every $x \in L$, it follows that $G^{-1}f(x) \in R(X)$.

From $f(L) \subseteq G(L)$ it follows that (i) is satisfied. Let us prove that $G^{-1}(f(S))$ is of Zima´s type. By $\mathcal{U}$ we shall denote the family of all neighbourhoods of zero in X and let $V \in \mathcal{U}$. We shall prove that there exists $U \in \mathcal{U}$ so that:

$$co(U \cap (G^{-1}f(S) - G^{-1}f(S))) \subseteq V .$$

Since G^{-1} is continuous and linear, there exists $V' \in \mathcal{U}$ so that $G^{-1}(V') \subseteq V$. Further, the set f(L) is of Zima´s type and so

there exists $U' \in \mathcal{U}$ such that:

$$co(U' \cap (f(S)-f(S))) \subseteq V' .$$

From the linearity of the mapping G^{-1}, we have that:

$$G^{-1}(co(U' \cap (f(S)-f(S)))) = co(G^{-1}(U' \cap (f(S)-f(S)))) .$$

This implies that:

$$(2) \quad co(G^{-1}(U') \cap G^{-1}(f(S)-f(S))) \subseteq G^{-1}(V') \subseteq V .$$

Further, the mapping G is continuous and so there exists $U \in \mathcal{U}$ such that $GU \subseteq U'$. Hence $U \subseteq G^{-1}(U')$, and from (2) we obtain that:

$$co(U \cap (G^{-1}f(S)-G^{-1}f(S))) \subseteq co(G^{-1}(U') \cap (G^{-1}f(S)-G^{-1}f(S))) \subseteq V.$$

Using the method of duality and Theorem A, in [11] the following fixed point theorem is proved:

THEOREM B. *Let* L *be a nonempty compact subset of Zima's type of a topological vector space* E, $f:L \to 2^E$ *an upper semicontinuous mapping such that* $L \subseteq f(L)$, $f(x) = \overline{f(x)}$ *for every* $x \in L$, $\overline{co} f^{-1}(x) = f^{-1}(x)$, *for every* $x \in f(L)$ *and* $f(L) = \overline{co}\ f(L)$ *be compact. Then there exists* $x_o \in L$ *such that* $x_o \in f(x_o)$.

Applying Theorem B we shall prove the following coincidence point theorem.

THEOREM 2. *Let* S *be a nonempty, compact and convex subset of Zima's type of tolopogical vector space* E, K *a compact subset of* E *such that* $S \subseteq K \subseteq E$, H *a convex subset of a topological vector space,* $\phi:S \to 2^H_c$, $\psi:K \to 2^H_{co}$ *(all convex subsets of* H*) upper semicontinuous mappings such that for every* $x \in S$:

$$\phi(x) \cap \psi(K) \neq \emptyset, \qquad \psi(x) \cap \phi(S) \neq \emptyset$$

and $\psi^{-1}(\phi(S)) = \overline{co}\psi^{-1}(\phi(S))$. *If for every* $x_1, x_2 \in S$ *and every* $\alpha, \beta \geq 0$, $\alpha+\beta = 1$, $\alpha\phi(x_1) + \beta\phi(x_2) \subseteq \phi(\alpha x_1+\beta x_2)$ *it follows that there exists* $x_o \in S$ *so that* $\phi(x_o) \cap \psi(x_o) \neq \emptyset$.

P r o o f. Let, as in Theorem 1:

$$\hat{\phi}(x) = \psi^{-1}(\phi(x)), \quad x \in S .$$

From the condition $\phi(x) \cap \psi(K) \neq \emptyset$, for every $x \in S$ it follows that $\hat{\phi}(x) \neq \emptyset$ for every $x \in S$. It is obvious that $\hat{\phi}(S) \subseteq K$. Let us prove that all the conditions of Theorem B for $L = S$ are satisfied.

First, let us prove that $L \subseteq \hat{\phi}(L)$. Since:

$$\psi^{-1}(\phi(S)) = \{x \mid x \in K, \ \psi(x) \cap \phi(S) \neq \emptyset\}$$

from $\psi(x) \cap \phi(S) \neq \emptyset$ for every $x \in S$, it follows that $S \subseteq \psi^{-1}(\phi(S))$ and so $L \subseteq \hat{\phi}(L)$. Furthermore, $\phi(x)$ is closed and ψ is upper semicontinuous and so $\hat{\phi}(x)$ is closed for every $x \in S$. It is obvious that $\hat{\phi}(S) = \overline{co}\hat{\phi}(S)$ and so it remains to be proved that:

$$\overline{co}\hat{\phi}^{-1}(x) = \hat{\phi}^{-1}(x), \quad \text{for every } x \in \hat{\phi}(S)$$

and that $\hat{\phi}$ is upper semicontinuous. The upper semicontinuity can be proved as in [15]. Since E is Hausdorff, $\{x\}$ closed and $\hat{\phi}$ upper semicontinuous, we conclude that $\hat{\phi}^{-1}(\{x\})$ is closed. We shall prove the convexity of $\hat{\phi}^{-1}(x)$, $x \in \hat{\phi}(S)$. Let $u_1, u_2 \in \hat{\phi}^{-1}(x)$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. This means that $\phi(u_1) \cap \psi(x) \neq \emptyset$, $\phi(u_2) \cap \psi(x) \neq \emptyset$. If $y_1 \in \phi(u_1) \cap \psi(x)$ and $y_2 \in \phi(u_2) \cap \psi(x)$, then $\alpha y_1 + \beta y_2 \in \psi(x)$, since $\psi(x)$ is convex and $\alpha y_1 + \beta y_2 \in \alpha\phi(u_1) + \beta\phi(u_2) \subseteq \phi(\alpha u_1 + \beta u_2)$.

So, we conclude that $\alpha y_1 + \beta y_2 \in \phi(\alpha u_1 + \beta u_2) \cap \psi(x)$.

3. EQUILIBRIUM STATE IN A SPECIAL NONCOOPERATIVE GAME

In this section we shall use Theorem A in order to obtain, similarly as in [12], a theorem on the equilibrium state in a special noncooperative game.

THEOREM 3. *Let* $\{E_i\}_{i \in I}$ *be a family of topological vector spaces, for each* $i \in I$, K_i *a closed and convex subset of*

E_i, $K = \prod_{i\in I} K_i$, $E = \prod_{i\in I} E_i$, *for each* $i \in I$, $\phi_i : K \to R(K_i)$ *be upper semicontinuous and* $\phi_i(K) \subseteq C_i \subseteq K_i$ $(i \in I)$ *where* C_i *is a compact set for each* $i \in I$. *If, for every* $i \in I$, $\phi_i(K)$ *is of Zima's type, there exists an* $\bar{x} \in K$ *such that* $\bar{x}_i \in \phi_i(\bar{x})$ *where* $\bar{x}_i = \mathrm{proj}_{K_i}\bar{x}$, $i \in I$.

P r o o f. The proof is similar to [12]. Let $\phi : K \to R(K)$ be defined by:

$$\phi(x) = \prod_{i\in I} \phi_i(x), \quad x \in K .$$

Then, $\phi(x)$ is compact, since it is the product of compact sets $\phi_i(x)$, $(i \in I)$. Furthermore, $\phi(x)$ is convex and so $\phi(x) \in R(K)$ for every $x \in K$. Since $\phi(K) \subseteq \prod_{i\in I} C_i$, the upper semicontinuity of ϕ follows from the closedness of the graph of ϕ. This can be proved as in [15]. It remains to be proved that $\phi(K)$ is of Zima's type. Let us denote by $\mathcal{V}$ the fundamental system of neighborhoods of zero in E and by $\mathcal{V}_i$ the fundamental system of neighbourhoods of zero in E_i $(i \in I)$. Let $V \in \mathcal{V}$. We shall show that there exists $U \in \mathcal{U}$ so that $\mathrm{co}(U \cap (\phi(K) - \phi(K))) \subseteq V$. Since $V \in \mathcal{V}$, there exists a finite set $J \subseteq I$ such that $V = \prod_{i\in I} V_i'$

where
$$V_i' = \begin{cases} E_i, & i \in I\setminus J \\ V_i, & i \in J \end{cases}$$
and $V_i \in \mathcal{V}_i$, $i \in J$.

Since $\phi_i(K)$ is of Zima's type, there exists $U_i \in \mathcal{V}_i$ $(i \in I)$ so that:

$$\mathrm{co}(U_i \cap (\phi_i(K) - \phi_i(K))) \subseteq V_i, \quad i \in J .$$

Let
$$U_i' = \begin{cases} U_i, & i \in J \\ E_i, & i \in I\setminus J \end{cases}$$
$(i \in I)$. Then it is easy to prove [12] that for $U = \prod_{i\in I} U_i'$ we have that $\mathrm{co}(U \cap (\phi(K) - \phi(K))) \subseteq V$.

As in [15] we can formulate the following theorem on the equilibrium state in a special noncooperative game.

THEOREM 4. *Let* $\{E_i\}_{i\in I}$ *be a family of topological vector spaces, for each* $i \in I$ *let* C_i *be a nonempty convex subset of* E_i, $C = \prod_{i\in I} C_i$, $\phi_i : C \to 2^{K_i}$ *be a continuous function for every* $i \in I$, *where* K_i *is a nonempty compact subset of* C_i *and* $f_i : C \to R^1$ *be a continuous function for every* $i \in I$. *If the set:*

$$(3) \quad \hat{\phi}_i(x) = \{y \mid y \in \phi_i(x), \quad f_i(y, x_i') = \max_{\hat{x}_i \in \phi_i(x)} f_i(\hat{x}_i, x_i')\}$$

is convex for each $x \in C$ *where,* $x_i' = \mathrm{proj}_{C_i'} x$, $C_i' = \prod_{j\in I, j\neq i} C_j$ $(i \in I)$ *and* $\phi_i(C)$ *is of Zima's type, then there exists an* $\bar{x} \in K = \prod_{i\in I} K_i$ *such that:*

$$(4) \quad f_i(\bar{x}) = \max_{\hat{x}_i \in \phi_i(\bar{x})} f_i(\hat{x}_i, \bar{x}_i') \text{ and } \bar{x}_i \in \phi_i(\bar{x}) .$$

P r o o f. The proof follows from Lemma 5 in [15] and Theorem 3.

COROLLARY 2. *Let* $\{(E_i, \|\ \|_i^*)\}_{i\in I}$ *be a family of paranormed spaces,* $C = \prod_{i\in I} C_i$, $\phi_i : C \to 2^{K_i}$ *be a continuous function for every* $i \in I$ *where* K_i *is a nonempty compact subset of nonempty convex subset* C_i *of* E_i, $f_i : C \to R^1$ *be a continuous function for every* $i \in I$ *so that* $\hat{\phi}_i(x)$ $(x \in C)$, *defined by (3), is convex. If there exists* M_i $(i \in I)$ *so that:*

$$\|tx\|_i^* \leq M_i t \|x\|_i^* , \quad t \in [0,1], \ x \in \phi_i(C) - \phi_i(C) ,$$

then there exists an $\bar{x} \in K = \prod_{i\in I} K_i$ *such that (4) holds.*

4. EXISTENCE THEOREMS FOR SOME CLASSES OF EQUATIONS

Using Corollary 1 we shall prove the following theorem.

THEOREM 5. *Let* X *be a topological vector space,* $\mathcal{U}$ *the fundamental system of zero in* X, K *a compact and convex subset of* X, G *a linear one to one mapping from* X *into* X *such that* G *and* G^{-1} *is continuous,* $T \in L(X,X)$ *and* S *an upper semi-continuous mapping from* K *into* $R(X)$ *such that the following two conditions are satisfied:*

(i) *For every* $y \in \overline{co}(S(K)-S(K))$ *there exists a unique* $x(y) \in G(K)$ *such that* $x(y) = Tx(y) + y$.

(ii) $0 \in G(K) \cap S(K)$ *and for every* $V \in \mathcal{U}$ *there exists* $U \in \mathcal{U}$ *so that* $co(U \cap (S(K)-S(K))) \subseteq V$.

Then there exists $x \in K$ *so that* $G(x) \in TG(x) + Sx$.

P r o o f. Let us define the mapping $R: \overline{co}(S(K)-S(K)) \to G(K)$ in the following way:

$$Ry = TRy + y, \text{ for every } y \in \overline{co}(S(K)-S(K)) .$$

We shall prove that the mapping R is continuous. Let $\{y_\alpha\}_{\alpha\in A}$ be a convergent net from $\overline{co}(S(K)-S(K))$ and $\lim\limits_{\alpha\in A} y_\alpha = y \in \overline{co}(S(K)-S(K))$. Since $\{Ry_\alpha\}_{\alpha\in A} \subseteq G(K)$ and $G(K)$ is compact, there exists a subnet $\{y_{\alpha_\beta}\}_{\beta\in B}$ such that $z = \lim\limits_{\beta\in B} Ry_{\alpha_\beta}$. Then from $Ry_{\alpha_\beta} = TRy_{\alpha_\beta} + y_{\alpha_\beta}$ we have $\lim\limits_\beta Ry_{\alpha_\beta} = \lim\limits_\beta TRy_{\alpha_\beta} + \lim\limits_\beta y_{\alpha_\beta}$ and so $z = Ry$. From this it is easy to conclude that $Ry = \lim\limits_\beta Ry_{\alpha_\beta}$. Furthermore, from $M \subseteq S(K)-S(K)$ it follows that $co\,R(M) = R(co\,M)$. Indeed, if $u \in coR(M)$, then:

$$u = \sum_{i=1}^{n} t_i u_i,\ u_i \in R(M),\ t_i \geq 0\ (i \in \{1,2,\dots,n\}),\ \sum_{i=1}^{n} t_i = 1.$$

Since $u_i \in R(M)$ $(i \in \{1,2,\ldots,n\})$, there exists $v_i \in M$ $(i \in \{1, 2,\ldots,n\})$ such that $u_i = Rv_i$ $(i \in \{1,2,\ldots,n\})$ and we have that:

$$Rv_i = TRv_i + v_i (i \in \{1,2,\ldots,n\}). \quad \text{Hence:}$$

$$\sum_{i=1}^{n} t_i Rv_i = T\left(\sum_{i=1}^{n} t_i Rv_i\right) + \sum_{i=1}^{n} t_i v_i, \quad \sum_{i=1}^{n} t_i v_i \in \operatorname{co} M$$

which implies that $R\left(\sum_{i=1}^{n} t_i v_i\right) = \sum_{i=1}^{n} t_i Rv_i$. So, we have that $R(\operatorname{co} M) = \operatorname{co} R(M)$.

Since $0 \in S(K)$, it follows that $S(K) \subseteq \overline{\operatorname{co}}(S(K)-S(K))$ and so we can define the mapping R^* in the following way: $R^*x = \bigcup_{y \in Sx} Ry$, for every $x \in K$. Since T is a linear mapping from X into X and for every $y \in \overline{\operatorname{co}}(S(K)-S(K))$ there exists one and only one element $x(y) \in G(K)$ such that $Ry = TRy + y$, it follows that $R(0) = 0$.

So, for every $V \in \mathcal{U}$ there exists $V' \in \mathcal{U}$ such that:

$$R(V' \cap \operatorname{co}(S(K)-S(K))) \subseteq V .$$

The rest of the proof is similar to [3], but we shall repeat it here for completeness. Namely, we shall prove that the mappings G and R^* satisfy all the conditions of Corollary 1 and that there exists $x \in K$ such that $G(x) \in R^*(x)$.

It is obvious that R^* is an upper semicontinuous mapping from K into $\mathcal{R}(X)$, since R is continuous and S is an upper semicontinuous mapping from K into $\mathcal{R}(X)$ and $M = S(x)$, $x \in K$ implies that $R(M)$ is convex. Furthermore, there exists $U' \in \mathcal{V}$ such that:

$$\operatorname{co}(U' \cap (S(K)-S(K))) \subseteq V' \cap \operatorname{co}(S(K)-S(K)) .$$

Since $R(\operatorname{co}(U' \cap (S(K)-S(K)))) = \operatorname{co}(R(U' \cap (S(K)-S(K))))$ it follows that $\operatorname{co}(R(U' \cap (S(K)-S(K)))) \subseteq V$. We have that $R^{-1}z = z - Tz$ for every $z \in R(\overline{\operatorname{co}}(S(K)-S(K)))$. Hence, R^{-1} is continuous, and so there exists $U \in \mathcal{U}$ such that:

$$R^{-1}(U\cap R(S(K)-S(K))) \subseteq U'\cap(S(K)-S(K)) \; .$$

Hence, $co(U\cap(R^*K-R^*K))\subseteq V$, since for every $x,y\in S(K)$, $R(x-y) = Rx-Ry$. From Corollary 1 it follows that there exists $x\in K$ such that $G(x)\in R^*(x)$.

This means that there exists $u\in S(x)$ such that $G(x) = Ru$. So, we have that $G(x) = Ru = TRu + u = TG(x) + u \in TG(x) + S(x)$.

REMARK. From the proof it is easy to conclude that it is sufficient to suppose that K is a nonempty closed and convex subset for X, that the set $\overline{\{x(y)\}}_{y\in\overline{co}(S(K)-S(K))}$ is compact and S such that the set $\overline{S(K)}$ is compact.

That is, in this case we can also prove that the mapping R is continuous.

COROLLARY 3. [3] *Let* X *be a topological vector space,* K *a compact, convex subset of* X, $O\in K$, $T\in L(X,X)$ *and* $S:K\to \mathcal{R}(X)$ *an upper semicontinuous mapping. Suppose that the following two conditions are satisfied:*

(i) *For every* $y\in\overline{co}(S(K)-S(K))$ *there exists one and only one element* $x(y)\in K$ *such that* $x(y) = Tx(y) + y$.

(ii) $O\in S(K)$ *and* $S(K)$ *is of Zima's type.*

Then there exists $x\in K$ *such that* $x\in Tx + Sx$.

P r o o f. It is enough to take in Theorem 5 that $Gx = x$, $x\in X$.

COROLLARY 4. *Let* $(X,\|\ \|^*)$ *be a complete paranormed space,* K *a closed and convex subset of* X, $S:K\to\mathcal{R}(X)$ *an upper semicontinuous mapping,* $T\in L(X,X)$, $\overline{S(K)}$ *compact,* $O\in S(K)\cap K$ *so that the following conditions are satisfied:*

1. $\|Tx\|^*\le q\|x\|^*$, *for every* $x\in X$, *where* $q\in[0,1)$
2. $T(K)+\overline{co}(S(K)-S(K))\subseteq K$
3. *There exists* $C>0$ *so that* $\|tz\|^*\le Ct\|z\|^*$ *for every* $t\in[0,1]$ *and every* $z\in S(K)-S(K)$.

Then, there exists $x\in K$ *such that* $x\in Tx+Sx$.

P r o o f. It remains to be proved that for every $y \in \overline{co}(S(K)-S(K))$ there exists one and only one element $x(y) \in K$ such that $x(y) = Tx(y) + y$ and the set $\{\overline{x(y)}\}_{y \in co(S(K)-S(K))}$ is compact. Since $\| Tx-Ty \|^* \leq q \| x-y \|^*$, for every $x,y \in X$ and

$$T(K) + \overline{co}(S(K)-S(K)) \subseteq K$$

it follows from the Banach fixed point theorem that for every $y \in \overline{co}(S(K)-S(K))$ there exists $x(y) \in K$ so that $x(y) = Tx(y) + y$. From the inequality:

$$\| x(y_1) - x(y_2) \|^* \leq \frac{\| y_1 - y_2 \|^*}{1-q}$$

for every $y_1, y_2 \in \overline{co}(S(K)-S(K))$ follows the continuity of the mapping $y \mapsto x(y)$ $(y \in \overline{co}(S(K)-S(K)))$. The rest of the proof is similar to the proof of Theorem 5.

THEOREM 6. *Let* K *be a nonempty, compact, convex subset of topological vector space* E*,* F *a topological vector space,* g *a continuous mapping of* $K \times K$ *into* F*, and* C *a closed subset of* F*. Suppose that for each* x *in* K *the set:*

$$\{y \mid y \in K,\ g(x,y) \in C\}$$

is nonempty and convex. If K *is of Zima's type, there exists an element* $u \in K$ *such that* $g(u,u) \in C$.

P r o o f. As in [1] for each x in K, we define T(x) in the following way:

$$Tx = \{y \mid y \in K,\ g(x,y) \in C\}.$$

Since $T(K) \subseteq K$ and K is of Zima's type, it follows that T(K) is of Zima's type. Furthermore, the mapping $T: K \to 2^K$ satisfies the condition $T(x) = \overline{co}T(x)$ for every $x \in K$, since C is closed and g continuous. The upper semicontinuity of T follows as in [1], and so from Theorem A it follows that there exists $x \in K$ such that $x \in Tx$. This implies that $g(x,x) \in C$.

As in [1], from Theorem 6 we obtain the following Corollaries.

COROLLARY 5. *Let* K *be a compact, convex subset of topological vector space* E, F *a topological vector space, and* g *a continuous mapping of* $K \times K$ *into* F. *Suppose that* $g(x, t_1y_1 + t_2y_2) = t_1g(x,y_1) + t_2g(x,y_2)$ *for all* $x, y_1, y_2 \in K$ *and* $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$ *and there exists for every* $x \in K$, $y \in K$ *such that* $g(x,y) = 0$. *If* K *is of Zima's type, there exists* $u \in K$ *such that* $g(u,u) = 0$.

P r o o f. For $C = \{0\}$, the set $\{y | y \in K,\ g(x,y) \in C\}$ is nonempty and convex for every $x \in K$.

COROLLARY 6. *Let* K *be a compact, convex subset of Zima's type of topological vector space* E, C *a nonempty, closed, convex subset of* E *such that* $f(K) \subseteq K + C$. *Then there exists an element* $u \in K$ *such that* $f(u) \in u + C$.

P r o o f. Let in Theorem 6, $F = E$ and $g(x,y) = f(x) - y$ for every $x, y \in K$. Then

$$\{y | y \in K,\ g(x,y) \in C\} = K \cap (f(x) - C)$$

and so the set $\{y | y \in K,\ g(x,y) \in C\}$ is nonempty and convex for every $x \in K$.

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REZIME

NEKE PRIMENE TEOREME O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA PRESLIKAVANJA U VEKTORSKO TOPOLOŠKIM PROSTORIMA

U ovom radu su dokazane teoreme o koincidenciji za višeznačna preslikavanja a data je i primena teoreme o nepokretnoj tački iz rada [3] u teoriji igara. Dokazane su i tri teoreme o postojanju rešenja nekih klasa jednačina u vektorsko topološkom prostoru.

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ON A COMMON FIXED POINT IN QUASI- - UNIFORMIZABLE SPACES

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ABSTRACT

Using a similar method as in [7] we prove in this paper a generalization of Fisher's fixed point theorem in quasi-uniformizable spaces. First, we shall give some difinitions from [5] and [7].

1. Let X be an arbitrary set, $\{d_i | i\in I\}$ be a family of mappings of XxX into $\mathbb{R}^+$ and $g: I\to I$.

DEFINITION 1. *A triplet* $(X, \{d_i\}_{i\in I}, g)$ *is said to be a quasi-uniformizable space if for every* $x,y,z\in X$ *and* $i\in I$ *we have:*

(a) $d_i(x,y)\geq 0, d_i(x,x)=0$,

(b) $d_i(x,y)=d_i(y,x)$,

(c) $d_i(x,y)\leq d_{g(i)}(x,z)+d_{g(i)}(z,y)$.

A quasi-uniformizable space $(X, \{d_i\}_{i\in I}, g)$ is Hausdorff if the relation $d_i(x,y)=0$, for every $i\in I$ implies $x=y$. A Hausdorff quasi-uniformizable space $(X, \{d_i\}_{i\in I}, g)$ becomes a Hausdorff topological space if the fundamental system of neighbourhoods of $x\in X$ is given by:

$$B(x;\varepsilon,i)=\{y\in X : d_i(x,y)<\varepsilon\}.$$

A mapping $t: [0,1]^2\to[0,1]$ is called a T-norm if for every $a,b,c,d\in[0,1]$:

1. $t(0,0)=0, t(a,1)=a$,
2. $t(a,b)=t(b,a)$,

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3. $t(a,b)\geq t(c,d)$ if $a\geq c$ and $b\geq d$,

4. $t(t(a,b),c)=t(a,t(b,c))$.

Now, let X be a vector space. I be an index set, L be a family of distribution functions and for every $i\in I$, $F^i: X\to L$.

DEFINITION 2. [5] *A triplet* $(X,\{F^i\}_{i\in I},t)$ *is called a probabilistic locally convex space if* t *is a* T-*norm and for every* $i\in I$ *the following conditions are satisfied* ($F^i_x = F^i(x)$):

A. $F^i_x(s) = 1$, *for every* $s>0 \Longleftrightarrow x=0$,

B. $F^i_x(0)=0$, *for every* $x\in X$,

C. $F^i_{rx}(s)=F^i_x\left(\frac{s}{|r|}\right)$, *for every* $x\in X, s>0$ *and* $r\in K\setminus\{0\}$ (K *is the scalar field*).

D. $F^i_{x+y}(s_1+s_2)\geq t(F^i_x(s_1),F^i_y(s_2))$, *for every* $x,y\in X$ *and every* $s_1,s_2>0$.

In X the (ε,λ)-topology is introduced in the following way:

The fundamental system of neighbourhoods of $x\in X$ is given by the family $U=\{U^i_x(\varepsilon,\lambda): i\in I,\ \varepsilon>0, \lambda\in(0,1)\}$ where $U^i_x(\varepsilon,\lambda)=\{y\in X: F^i_{x-y}(\varepsilon)>1-\lambda\}$.

In [7] it is shown that a probabilistic locally convex space $(X,\{F^i\}_{i\in I},t)$, such that $\sup\limits_{a<1} t(a,a)=1$, is a quasi-uniformizable space in which the family $\{d_j\}_{j\in I'}$, is defined in the following way:

For $j=(i,\lambda)\in I'$, where $i\in I$ and $\lambda\in(0,1)$, $d_j(x,y)= =\sup\{s: F^i_{x-y}(s)\leq 1-\lambda\}$.

The construction of the mapping $g: I'\to I'$ is as follows. From $\sup\limits_{a<1} t(a,a)=1$ it follows that for every $\lambda\in(0,1)$ the exists $\delta_\lambda\in(0,1)$ so that for every $\delta\leq\delta_\lambda$, $t(1-\delta,1-\delta)\geq 1-\frac{\lambda}{2}$.

Let $\bar{g}(\lambda)=\sup\{\delta_\lambda$: where δ_λ is defined above $\}$.

Then $g(j)=(i,\bar{g}(\lambda))$, for $j=(i,\lambda)$.

2. Now, we shall give a generalization of a common fixed point theorem from [1] in quasi-uniformizable spaces. This theorem is also a generalization of Theorem 1 from [4].

THEOREM *Let* $(X,\{d_i\}_{i\in I},g)$ *be a sequentially complete Hausdorff quasi-uniformizable space,* $f:I\to I$, S *and* T *be continuous mappings from* X *into* X, $A:X\to SX\cap TX$ *be continuous so that* A *commutes with* S *and* T *and the following conditions are satisfied:*

1. *For every* $i\in I$, *there exists* $q_i:\mathbb{R}^+\to[0,1]$, *which is a nondecreasing function for which is* $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(t)<1$ *for every* $i\in I$ *and every* $t\in\mathbb{R}^+$ *and:*
   $d_i(Ax,Ay)\le q_i(d_{f(i)}(Sx,Ty))d_{f(i)}(Sx,Ty)$
   for every $i\in I$ *and every* $x,y\in X$.
2. *There exists* $x_o\in X$ *so that for every* $i\in I$:
   $$\sup_{j\in O(i,f),\,p\in N} d_j(Ax_o,Ax_p)=K_i\in\mathbb{R}^+$$
   where $O(i,f)=\{i,f(i),f^2(i),\ldots\}$ *and* $\{x_p\}_{p\in N}$ *is defined by:* $Sx_{2n-1}=Ax_{2n-2}$, $Ax_{2n-1}=Tx_{2n}$ $(n\in N)$.

Then there exists $z\in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i\in I$:

$$(1)\quad \sup_{j\in O(i,f)} d_j(A^3x_1,A^2x_o)=M_i\in\mathbb{R}^+$$

then Az *is a common fixed point for* A,S *and* T. *Further, let* $M=\{w:w\in X, w=Aw=Sw=Tw,$ *there exists* $\{R_i\}_{i\in I}$, *so that for every* $i\in I:\sup_{j\in O(i,f)} d_j(Az,w)\le R_i\}$. *Then* $M=\{Az\}$.

Proof: Similarly as in [4] it is easy to prove that for every $k\in N$ and $i\in I$:

$$d_i(Ax_{2k},Ax_{2k-1}) \leq \prod_{s=0}^{2k-2} q_{f^s(i)}(K_i)K_i$$

$$d_i(Ax_{2k+1},Ax_{2k}) \leq \prod_{s=0}^{2k-1} q_{f^s(i)}(K_i)K_i \; .$$

Since $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(K_i) \leq Q_i < 1$, for every $i \in I$, it follows that there exists $n_i \in N$ so that $q_{f^n(i)}(K_i) \leq Q_i$, for every $n \geq n_i$ which implies that:

$$d_i(Ax_n,Ax_{n-1}) \leq S_i Q_i^n \; , \text{ for every } i \in I, \quad n \in N.$$

Let us prove that $\{Ax_n\}_{n \in N}$ is a Cauchy sequence which means that for every $i \in I$ and every $\varepsilon > 0$ there exists $n(i,\varepsilon) \in N$ so that $d_i(Ax_n,Ax_{n+p}) < \varepsilon$, for every $n \geq n(i,\varepsilon)$, $p \in N$. Let $m \geq k$, $i \in I$. From the definition of the sequence $\{x_n\}_{n \in N}$ and condition 1 of the Theorem it follows that:

$$d_i(Ax_{2k},Ax_{2m+1}) \leq q_i(d_{f(i)}(Ax_{2m},Ax_{2k-1}))\ldots q_{f^{2k-1}(i)}($$

$$(d_{f^{2k}(i)}(Ax_o,Ax_{2m+1-2k}))d_{f^{2k}(i)}(Ax_o,Ax_{2m+1-2k}))$$

and similarly for $2k > 2m+1$:

$$d_i(Ax_{2k},Ax_{2m+1}) \leq q_i(d_{f(i)}(Ax_{2m},Ax_{2k-1}))\ldots q_{f^{2m}(i)}($$

$$(d_{f^{2m+1}(i)}(Ax_o,Ax_{2k-2m-1}))d_{f^{2m+1}(i)}(Ax_o,Ax_{2k-2m-1})).$$

Using condition 2 and the property of q_i that $q_i(t) \leq 1$ for every $t \in \mathbb{R}^+$ we obtain that for every $i \in I$:

$$d_i(Ax_{2k},Ax_{2m+1}) \leq q_i(K_i)\ldots q_{f^{2k-1}(i)}(K_i)K_i \; (m \geq k)$$

and :

$$d_i(Ax_{2k},Ax_{2m+1}) \leq q_i(K_i)\ldots q_{f^{2m}(i)}(K_i)K_i \; (2k > 2m+1).$$

This implies that :

$$d_i(Ax_n,Ax_{n+p}) \leq q_i(K_i)\ldots q_{f^{n-1}(i)}(K_i)K_i$$

for every $i \in I$ and for $n=2k, p=2m+1$ or $n=2k+1$, $p=2m+1$. Let $p=2m$ and $n=2k$ or $n=2k+1$. Then:

$$d_i(Ax_n, Ax_{n+p}) \le d_{g(i)}(Ax_n, Ax_{n+1}) + d_{g(i)}(Ax_{n+1}, Ax_{n+1+p-1}) \le$$

$$\le \prod_{s=0}^{n-1} q_{f^s(g(i))}(K_{g(i)})K_{g(i)} + \prod_{s=0}^{n} q_{f^s(g(i))}(K_{g(i)})K_{g(i)}.$$

Since $\overline{\lim\limits_{s\to\infty}}\ q_{f^s(g(i))}(K_{g(i)})<1$, for every $i \in I$ it follows that $\{Ax_n\}_{n \in N}$ is a Cauchy sequence. Let $z=\lim\limits_{n\to\infty} Ax_n$. As in [4] it follows that $Az=Sz=Tz$.

Further (1) implies that Az is a common fixed point for A,S and T. Indeed, from $d_j(A^2z, Az)=\lim\limits_{n\to\infty} d_j(A^2Ax_{2n+1}, AAx_{2n})$ for every $j \in O(i,f)$. using condition 1.and (1) we conclude that $d_j(A^2z,Az) \le M_i$, for every $j \in O(i,f)$, since for every $n \in N$:

$$d_j(A^3x_{2n+1}, A^2x_{2n}) \le d_{f^{2n}(j)}(A^3x_1, A^2x_o) \le M_i, \text{ for every } i \in I,$$

where $j \in O(i,f)$.

So we have that:

$$d_i(A^2z,Az) \le q_i(M_i)q_{f(i)}(M_i)\ldots q_{f^n(i)}(M_i)M_i$$

for every $i \in I$ which implies that $d_i(A^2z,Az)=0$, for every $i \in I$, This implies that Az is a common fixed point for A,S and T.

Let us prove that $M=\{Az\}$. Suppose that $w=Aw=Tw=Sw$ and for every $i \in I$:

$$\sup_{j\in O(i,f)} d_j(Az,w) \le R_i .$$

Then :

$$d_i(Az,w) = d_i(A(Az),Aw) \le q_i(d_{f(i)}(S(Az),Tw))d_{f(i)}(S(Az),Tw)=$$

$$=q_i(d_{f(i)}(Az,w))d_{f(i)}(Az,w) \le \ldots \le q_i(d_{f(i)}(Az,w))\ldots \times$$

$$q_{f^n(i)}(d_{f^{n+1}(i)}(Az,w))d_{f^{n+1}(i)}(Az,w).$$

From this it follows that $d_j(Az,w) \leq R_i$ $(j \in O(i,f))$ and so $d_i(Az,w) \leq q_i(R_i) q_{f(i)}(R_i) \dots q_{f^n(i)}(R_i) R_i$.

Since $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(R_i) < 1$ it follows that $d_i(Az,w) = 0$,

for every $i \in I$ and so $Az=w$.

Remark: If there exists $u \in X$ so that for every $i \in I$:

$$d_j(Az,u) \leq T_i, \text{ for every } j \in O(i,f)$$

and $g: O(i,f) \to O(i,f)$, for every $i \in I$, then there exists one and only one common fixed point $w \in X$ for A,S and T so that for every $i \in I$:

$$d_j(w,u) \leq T'_i \text{ , for every } j \in O(i,f).$$

Namely, then we have :

$$d_j(Az,w) \leq d_{g(i)}(Az,u) + d_{g(i)}(u,w) \leq T_i + T'_i \text{ , for every}$$

$i \in I$ and every $j \in O(i,f)$ and in the Theorem is proved that $Az=w$.

COROLLARY 1. *Let* $(X,\{d_i\}_{i \in I})$ *be a sequentially complete Hausdorff uniformizable space,* $f: I \to I$, *S and T be continuous mappings from X into X,* $A: X \to SX \cap TX$ *be continuous so that condition 1. of the Theorem is satisfied and there exists* $x_o \in X$ *and* $x_1 \in X$ *so that* $Sx_1 = Ax_o$ *and for every* $i \in I$:

$$\sup_{n\in N} d_{f^n(i)}(Ax_o, Ax_1) = K_i, K_i \in \mathbb{R}^+$$

Then there exists $z \in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i \in I$: $\sup\limits_{n\in N} d_{f^n(i)}(A^3x_1, A^2x_o) = M_i$, $M_i \in \mathbb{R}^+$ *then Az is a common fixed point for A,S and T. Further, if for every* $i \in I$:

$$\sup_{n\in N} d_{f^n(i)}(A^2x_1, A^2x_o) = R_i, \; R_i \in \mathbb{R}^+ \tag{2}$$

then there exists one and only one element $w \in X$ *such that*

(3) $\sup_{n\in N} d_{f^n(i)}(w,A^2x_o)=N_i, N_i \in \mathbb{R}^+$, *for every* $i\in I$

and $Aw = Sw = Tw = w$.

P r o o f: Every uniformizable space $(X,\{d_i\}_{i\in I})$ is a quasi-uniformizable space, where $g(i)=i$, for every $i\in I$. So we have that:

$$d_i(Ax_p,Ax_o)\le d_i(Ax_p,Ax_{p-1})+d_i(Ax_{p-1},Ax_{p-2})+\ldots+d_i(Ax_1,Ax_o)$$

(for every $i\in I, p\ge 2$). Since for every $j\in O(i,f)$

$$d_j(Ax_p,Ax_{p-1})\le \prod_{s=0}^{p-2} q_{f^s(j)}(K_i)K_i,$$ for every $i\in I$, it follows that :

$$d_j(Ax_p,Ax_o)\le \sum_{r=1}^{p} \left(\prod_{s=0}^{r-2} q_{f^s(j)}(K_i)\right)K_i.$$

Since $\overline{\lim}_{n\in N} q_{f^n(i)}(K_i)<1$ there exists $n_i\in N$ so that :

$$q_{f^n(i)}(K_i)\le Q_i<1, \text{ for every } n\ge n_i$$

and if $j\in\{f^s(i)\mid s\ge n_i\}$ then:

$$d_j(Ax_p,Ax_o)\le \sum_{r=1}^{\infty} Q_i^{r-1} K_i, \text{ for every } p\ge 2.$$

Since $Q_i<1$ it is easy to see that condition 2.of the Theorem is satisfied. So, there exists $z\in X$ such that Az is a common fixed point for A,S and T. From (2) it follows that $d_j(Az,A^2x_o)\le R_i$, for every $i\in I$ and every $j\in O(i,f)$ and using the Remark we conclude that there exists one and only one element $w\in X$ such that w is a common fixed point for A, S and T and that (3) is satisfied.

Using the Theorem we obtain the following corollory which is a generalization of the Theorem 1 from [2] .

COROLLARY 2 *Let* $(X,\{F^i\}_{i\in I},t)$ *be a sequentially complete Hausdorff probabilistic locally convex space where*

$\sup_{a<1} t(a,a)=1$, $f:I\to I$, S *and* T *be continuous mappings from* X *into* X, $A:X\to SX\cap TX$ *be a continuous mapping which commutes with* S *and* T *and the following conditions are satisfied:*

(i) *For every* $i\in I$, *there exists* $q_i:\mathbb{R}^+\to[0,1]$, *which is a nondecreasing function continuous from the right such that for every* $i\in I$ *and every* $s\in\mathbb{R}^+$ $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(s)<1$ *and for every* $i\in I$, *every* $x,y\in X$ *and every* $s\in\mathbb{R}^+$:

$$F^{i}_{Ax-Ay}(q_i(s)s)\geq F^{f(i)}_{Sx-Ty}(s)$$

(ii) *There exists* $x_o\in X$ *so that for every* $i\in I$: $\overline{\lim_{s\to\infty}}\, F^{j}_{Ax_o-Ax_p}(s)=1$, *uniformly in* $j\in O(i,f)$ *and* $p\in N$, *where* $\{x_p\}_{p\in N}$ *is defined by* $Sx_{2n-1}=Ax_{2n-2}$, $Tx_{2n}=Ax_{2n-1}$ *for every* $n\in N$.

Then there exists $z\in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i\in I$, $\overline{\lim_{s\to\infty}}\, F^{j}_{A^3x_1-A^2x_o}(s)=1$, *uniformly in* $j\in O(i,f)$ *then* Az *is a common fixed point for* A,S *and* T. *Further, let* $M'=\{w: w\in X, w=Aw=Sw=Tw$, *for every* $i\in I$, $\lim_{s\to\infty} F^{j}_{Az-w}(s)=1$, *uniformly in* $j\in O(i,f)\}$. *Then* $M'=\{Az\}$.

P r o o f: As in [7] it follows that (i) and (ii) implies 1. and 2. from the Theorem and that (1) is satisfied since for every $i\in I$, $\lim_{s\to\infty} F^{j}_{A^3x_1-A^2x_o}(s)=1$, uniformly in $j\in O(i,f)$, where $d_j(x,y)=\sup\{s: F^{i}_{x-y}(s)\leq 1-\alpha\}$, for every $j=(i,\alpha)\in I'$, $i\in I$ and $\alpha\in(0,1)$. Since, $w\in M'$ implies $w\in M$, where M is from the Theorem, it follows $M'=\{Az\}$.

Remark: If AX is a probabilistic bounded subset of $SX\cap TX$, (i) from Corollary 2 is satisfied and for every $i\in I$ there exists $h(i)\in I$ such that:

$$F_x^{f^n(i)}(s)\geq F_x^{h(i)}(s),\text{ for every } s>0,\text{ every } x\in X \text{ and}$$

every $n\in N$ it is easy to see that there exists one and only one element $x\in X$ such that Ax is the unique common fixed point for A,S and T.

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REZIME

O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U KVAZI-UNIFORMIZABILNIM PROSTORIMA

U ovom radu su uopšteni rezultati rada [4].

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ON THE NUMBER OF ABELIAN GROUPS OF A GIVEN ORDER AND THE NUMBER OF PRIME FACTORS OF AN INTEGER

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ABSTRACT

Let $a(n)$ and $\omega(n)$ denote the number of non--isomorphic abelian groups with n elements and the number of distinct prime factors of n respectively. The distribution of values of $a(n)$ (which is multiplicative) and $\omega(n)$ (which is additive) is compared in several ways.

1. INTRODUCTION

Let, as usual, $a(n)$ denote the number of non--isomorphic abelian groups with n elements. It is well--known (see [3]) that $a(n)$ is a multiplicative function of n such that $a(p^k) = P(k)$ for every prime p and every natural number k, where $P(k)$ is the number of unrestricted partitions of k (here and later $p, p_1, p_2, \ldots$ denote primes). Various problems concerning the distribution of values of $a(n)$ and related multiplicative functions were investigated in [4], [5] and [8]. Thus for

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instance it was proved in [4] that for $k \geq 1$ fixed

$$A_k(x) = \sum_{n \leq x, a(n)=k} 1 = d_k x + O(x^{1/2} \log x) \tag{1.1}$$

holds uniformly in k ($f(x) = O(g(x))$ and $f(x) \ll g(x)$ both mean $|f(x)| < Cg(x)$ for $x \geq x_0$ and some constant $C > 0$). The non-negative constant d_k is called the local density of $a(n)$, and as shown in [5] it satisfies the inequality

$$d_k \leq c_1 \exp(-c_2 \log k \cdot \log\log k), \quad (k \geq 3) \tag{1.2}$$

with some $c_1, c_2 > 0$. From the product representation

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\ldots \quad (\mathrm{Re}\ s > 1) \tag{1.3}$$

it is seen that the mean value of $a(n)$ equals $\zeta(2)\zeta(3)\zeta(4)\ldots = 2.29485\ldots$. Thus $a(n)$ is small on the average, although one has (see [7])

$$\limsup_{n\to\infty} \log a(n) \log\log n/(\log n) = (\log 5)/4, \tag{1.4}$$

and the bound implied by (1.4) is asymptotically attained for $n = (p_1 p_2 \ldots p_k)^4$, where p_i is the i-th prime. It seemed interesting to compare the values of $a(n)$ and other common arithmetical functions such as $d(n)$ and $\omega(n)$, which represent the number of divisors and the number of distinct prime factors of n respectively. From the elementary formulas

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}) \quad (\gamma = 0.577\ldots) \tag{1.5}$$

and

$$\sum_{n \leq x} \omega(n) = x \log\log x + Bx + O(x/\log x), \quad (B > 0) \tag{1.6}$$

it is seen that the average order of $d(n)$ and $\omega(n)$ is $\log n$ and $\log\log n$ respectively.
Therefore it is no surprise that

$$(1.7) \qquad \sum_{n\le x,\, d(n)>a(n)} 1 = x + O(x\log^{\varepsilon-1}x),$$

$$(1.8) \qquad \sum_{n\le x,\, \omega(n)>a(n)} 1 = x + O(x(\log\log x)^{-K}).$$

These formulas were proved in [5], and here $0 < \varepsilon < 1$, while K is an arbitrary, but fixed positive number.

2. STATEMENT OF RESULTS

In this note we shall further compare the values of $a(n)$ and $\omega(n)$. Possible generalizations to other arithmetical functions which behave "similarly" as $a(n)$ and $\omega(n)$ will be omitted to make the exposition clearer; e.g. functions of class F_1 of [5] can be obviously considered instead of $a(n)$ only, and likewise instead of $\omega(n)$ one may consider the familiar function $\Omega(n)$, the number of all prime factors of n etc. The problem to compare the values of $a(n)$ and $\omega(n)$ seems interesting, because $a(n)$ is multiplicative ($a(mn) = a(m)a(n)$ for $(m,n) = 1$) and $\omega(n)$ is additive ($\omega(mn) = \omega(m) + \omega(n)$ for $(m,n) = 1$). In treating two multiplicative or two additive functions, one can make use of the fact that the product (or quotient) of two multiplicative functions is again multiplicative, while the sum (or difference) of two additive functions is again additive. However, in our case special methods have to be used which will simultaneously deal with $a(n)$ and $\omega(n)$. The first result is an improvement of (1.8), which we formulate as

THEOREM 1. *There is a constant* $C > 0$ *such that*

$$(2.1) \qquad \sum_{n\le x,\, \omega(n)>a(n)} 1 = x + O(x\exp(-C\log_3 x\log_4 x)).$$

Here we used the abbreviation $\log_r x = \log(\log_{r-1}x)$, $\log_1 x = \log x$ = the natural logarithm of x. At this point it

may be remarked that the equation $a(n) = \omega(n)$ holds for many n, and quantitatively we have, for any fixed $A > 0$,

$$\sum_{n\le x,\,\omega(n)=a(n)} 1 \gg x(\log\log x)^A/\log x \,. \tag{2.2}$$

To see this recall that for an infinity of integers $k \ge 5$ there exists an integer $r < k$ such that $P(r) = k$. Let $m = p_1p_2\ldots p_{k-1}p_k^r$, where the p_i's are distinct primes. Then $\omega(m) = k = P(r) = a(m)$, hence for k fixed

$$\sum_{n\le x,\,\omega(n)=a(n)} 1 \ge \sum_{m\le x} 1 \gg x(\log\log x)^{k-2}/\log x, \tag{2.3}$$

by a classical result of E. Landau (see p. 168 of [6]) concerning the number of n not exceeding x for which $\omega(n) = k$. Now (2.2) follows from (2.3), since the values of the partition function tend quickly to infinity because

$$P(k) = (1 + o(1))(4\sqrt{3}k)^{-1}\exp(\pi(2k/3)^{1/2}), \quad (k \to \infty) \tag{2.4}$$

by a classical result of G.H. Hardy and S. Ramanujan ([9], p. 240).

The next result shows that the equation $\omega(n) = ra(n)$ has many solutions for any real number $r > 0$. The result is

THEOREM 2. *Every real number* $r \ge 0$ *is the limit point of the sequence* $\omega(n)/a(n)$.

Finally we present an asymptotic formula for a sum involving the functions $\omega(n)$ and $a(n)$. This is

THEOREM 3. *There is a constant* $A > 0$ *such that*

$$\sum_{n\le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 = Ax\log\log x + O(x) \,. \tag{2.5}$$

As a corollary it follows that for almost all n we have

$$|\omega(n) - \log\log n| \le a(n)(\log\log n)^{1/2+\delta} \tag{2.6}$$

for any $0 < \delta < 1/2$. To see this, let $F(\delta,x)$ denote the number of $n \le x$ for which (2.6) fails to hold. Then, using (2.5), we obtain

$$F(\delta,x) \le \sum_{n\le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 (\log\log n)^{-1-2\delta} <<$$

$$x^{1/2+\varepsilon} + \sum_{\sqrt{x}<n\le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 (\log\log x)^{-1-2\delta} << x(\log\log x)^{-2\delta},$$

since $\log\log n = \log\log x + O(1)$ for $\sqrt{x} < n \le x$. The last expression above is $o(x)$ for $\delta > 0$ as $x \to \infty$, which justifies the claim that (2.6) holds for "almost all" n. Theorem 3 could be generalized by replacing $a(n)$ by $a^k(n)$ for any fixed integer $k \ge 1$, in which case the constant A in (2.5) would depend on k.

3. PROOFS OF THE THEOREMS

To prove (2.1) let $S(x)$ denote the number of $n \le x$ for which $\omega(n) \le a(n)$. Write

$$S(x) = \sum_{n\le x,\,\omega(n)\le a(n)} 1 = S_1 + S_2 \tag{3.1}$$

say, where in S_1 we sum over relevant n for which $a(n) \le \frac{1}{3}\log_2 x$, while in S_2 we sum over relevant n for which $a(n) > \frac{1}{3}\log_2 x$. Using a result of P. Erdös and J.-L. Nicolas [2] on the distribution of values of $\omega(n)$ we have

$$S_1 \le \sum_{n\le x,\,\omega(n)\le(\log_2 x)/3} 1 << x\log^{-c} x \tag{3.2}$$

for some $0 < c < 1$ (the exact value of c is unimportant here). To bound S_2 we use (1.1), (1.2) and (1.4) to obtain

$$S_2 \le \sum_{(\log_2 x)/3\le k\le \exp(2\log x/\log_2 x} (d_k x + O(x^{1/2}\log x)) \tag{3.3}$$

$$\ll x \sum_{k \geq (\log_2 x)/3} \exp(-c_2 \log k \log_2 k) + O(x^{1/2+\varepsilon})$$

$$\ll x \exp(-c_3 \log_3 x \log_4 x).$$

Combining (3.1), (3.2) and (3.3) it follows that

$$\sum_{n \leq x, \omega(n) > a(n)} 1 = [x] - S(x) =$$

$$x + O(x \exp(-C \log_3 x \log_4 x)),$$

as asserted.

For the proof of Theorem 2 we may consider $r > 0$ only, since for $n_k = (p_1 p_2 \cdots p_k)^2$ we have

$$\lim_{k\to\infty} \omega(n_k)/a(n_k) = \lim_{k\to\infty} k 2^{-k} = 0.$$

Suppose then that $r > 0$ and $0 < \varepsilon < r/2$ are given. Using (2.4) it is seen that we may find an integer $u > 1$ such that

$$(r - \varepsilon) P^m(u) \geq m \tag{3.4}$$

for $m \geq m_o$. Further for $m \geq m_o$ there exists an integer $k = k(m, r, \varepsilon)$ such that

$$k - 1 < (r - \varepsilon) P^m(u) \leq k,$$

and in view of (3.4) $k \geq m$. We must have $k \leq rP^m(u)$, since otherwise

$$rP^m(u) < k < 1 + (r - \varepsilon) P^m(u) ,$$

implying $\varepsilon P^m(u) < 1$, which is impossible for m large enough. Therefore

$$(r - \varepsilon) P^m(u) \leq k \leq rP^m(u), \tag{3.5}$$

and taking $n_m = (p_1 p_2 \cdots p_m)^u p_{m+1} \cdots p_{k-1} p_k$ we have

$$r - \varepsilon \le kP^{-m}(u) = \omega(n_m)/a(n_m) \le r ,$$

which proves Theorem 2, since ε may be arbitrarily small and we make $m \to \infty$.

The idea for a result like (2.5) originates with P. Turán [10], who proved

$$(3.6) \qquad \sum_{n \le x} (\omega(n) - \log\log n)^2 \ll x\log\log x$$

thus providing a simple proof of the classical result of Hardy & Ramanujan ([9], pp. 262-275) that almost all integers have about $\log\log n$ distinct prime factors. Therefore our asymptotic formula (2.5) may be considered as a "weighted" analogue of (3.6). Squaring out the expression on the left of (2.5) it is seen that the proof will follow from

$$(3.7) \qquad \sum_{n \le x} \left(\frac{\log\log n}{a(n)}\right)^2 = Bx(\log\log x)^2 + C_1 x\log\log x + O(x)$$

$$(3.8) \qquad \sum_{n \le x} \frac{\omega(n)\log\log n}{a^2(n)} = Bx(\log\log x)^2 + C_2 x\log\log x + O(x)$$

$$(3.9) \qquad \sum_{n \le x} \frac{\omega^2(n)}{a^2(n)} = Bx(\log\log x)^2 + C_3 x\log\log x + O(x)$$

where C_1, C_2, C_3 are suitable constants and

$$(3.10) \qquad B = \prod_p \{1 + \sum_{j=2}^{\infty} (P^{-2}(j) - P^{-2}(j-1))p^{-j}\}.$$

To prove (3.7)-(3.10) we proceed similarly as in the proof of (9.9) in [1].

Note that $a^{-2}(n)z^{\omega(n)}$ is a multiplicative function of n, so that for $\mathrm{Re}\, s > 1$ and $|z| \le 1$

$$\sum_{n=1}^{\infty} a^{-2}(n)z^{\omega(n)}n^{-s} = \prod_p (1 + zp^{-s} + 2^{-2}zp^{-2s} + 3^{-2}zp^{-3s} + 5^{-2}zp^{-4s} + \dots)$$

$$= \zeta^z(s)G(s,z),$$

where

$$G(s,z) = \prod_p (1 - p^{-s})^z (1 + zp^{-s} + 2^{-2}zp^{-2s} + 3^{-2}zp^{-3s} + 5^{-2}zp^{-4s} + \ldots$$

is absolutely and uniformly convergent for $\mathrm{Re}\, s > 1/2$ and $|z| \le C$ for any fixed $C > 0$. Using a well-known convolution result of A. Selberg ([1], Lemma 2.1) it follows that

$$(3.11) \qquad \sum_{n \le x} a^{-2}(n) z^{\omega(n)} = \frac{G(1,z)}{\Gamma(z)} x \log^{z-1} + R(x,z) ,$$

where uniformly for $|z| \le 3/2$ we have $R(x,z) \ll \ll x(\log x)^{\mathrm{Re}\, z-2} \ll x\log^{-1/2} x$. Differentiating (3.11) with respect to the complex variable z we obtain, for $|z| \le 1$,

$$(3.12) \qquad \sum_{n \le x} a^{-2}(n)\omega(n) z^{\omega(n)-1} = \frac{d}{dz}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x + $$

$$+ \frac{G(1,z)}{\Gamma(z)} x\log^{z-1} x\log\log x + O(x\log^{-1/2} x),$$

$$(3.13) \qquad \sum_{n \le x} a^{-2}(n)\omega(n)(\omega(n) - 1) z^{\omega(n)-2} = \frac{d^2}{dz^2}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x + $$

$$+ 2\frac{d}{dz}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x\log\log x + $$

$$+ \frac{G(1,z)}{\Gamma(z)} x\log^{z-1} x(\log\log x)^2 + O(x\log^{-1/2} x) ,$$

where we used Cauchy's inequality for derivatives of analytic functions to bound $\frac{\partial^k}{\partial z^k} R(x,z)\ (k=1,2)$. Setting $z = 1$ in (3.11)-(3.13), adding (3.12) to (3.13) and observing that $\frac{G(1,1)}{\Gamma(1)} = B$ (as defined by (3.10)), we obtain (3.7)-(3.9) by partial summation or by using $\log\log n = \log\log x + O(1)$ for $\sqrt{x} \le n \le x$. Finally a calculation shows that $A = C_1 - 2C_2 + C_3 > 0$, completing the proof of (2.5). In concluding it may be mentioned that the asymptotic formula (2.5) could be further sharpened (by introducing new main terms in place of $O(x)$) by using the methods developed in Ch. 5 of [1].

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REZIME

O BROJU ABELOVIH GRUPA DATOG REDA I BROJU PROSTIH FAKTORA CELOG BROJA

Neka $a(n)$ i $\omega(n)$ označavaju broj neizomorfnih Abelovih grupa sa n elemenata i broj različitih prostih faktora od n respektivno. Raspodela vrednosti $a(n)$ (koja je multiplikativna) i $\omega(n)$ (koja je aditivna) je uporedjena na nekoliko načina.

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THE APPROXIMATE SOLUTION OF A DIFFERENTIAL EQUATION IN MANY STEPS

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ABSTRACT

In this paper we construct the approximate solution of a certain linear partial differential equation with constant coefficients, using the field of Mikusiński operators M, in steps, on the interval $[0,T]$. We also give the error of approximation.

In papers [1] and [2] the following linear partial differential equation with constant coefficients was observed:

$$(1) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda,t)}{\partial\lambda^{\mu}\,\partial t^{\nu}} = \phi(\lambda,t), \quad \lambda_1 \le \lambda \le \lambda_2, \quad 0 \le t < \infty .$$

Using the field of Mikusiński operators M the approximate solution of equation (1) was constructed in [1] and the error of approximation was given in [2] on the interval $[0,T]$. This error of approximation increases rather fast by enlarging T.

In this paper we shall find the approximate solution of (1) for $n=1$ and $\phi(\lambda,t) = 0$, i.e.

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$$(2) \qquad \sum_{\mu=o}^{m} \sum_{\nu=o}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda,t)}{\partial\lambda^{\mu}\,\partial t^{\nu}} = 0$$

with the following conditions:

$$(3) \qquad \frac{\partial^{\mu} x(\lambda,0)}{\partial\lambda^{\mu}} = 0, \quad \lambda > 0,\ \mu = 0,\dots,m$$

$$(4) \qquad \frac{\partial^{\mu} x(0,t)}{\partial\lambda^{\mu}} = 0, \quad t > 0,\ \mu = 0,\dots,m-2$$

$$\frac{\partial^{m-1} x(0,t)}{\partial\lambda^{m-1}} = 1, \quad t > 0$$

on the interval $[0,T]$. We shall divide this interval into two intervals ($[0,T_1]$ and $[T_1,T]$)and seek the approximate solution succesively. We shall prove that this enables us to obtain a better estimation of the error. At the same time we can construct the approximate solution on an arbitrary interval $[T_1,T]$, $0 < T_1 < T$. The method which we shall use can be applied in many steps in the same way as in two steps (dividing the interval $[0,T]$ into intervals $[0,T_1]\,[T_1,T_2]\dots[T_{n-1},T_n]$ where $T_n = T$).

In the field M the differential equation

$$(5) \qquad \sum_{\mu=o}^{m} \sum_{\nu=o}^{1} \alpha_{\mu,\nu} s^{\nu} x^{(\mu)}(\lambda) = f(\lambda)$$

where

$$(5') \qquad f(\lambda) = \sum_{\mu=o}^{m} \alpha_{\mu,1} \frac{\partial^{\mu} x(\lambda,t)}{\partial\lambda^{\mu}} \Big|_{t=o}$$

corresponds to equation (2). Conditions (3) imply $f(\lambda) = 0$. The exact solution of equation (5) with (5´) has the form ([1]):

$$(6) \qquad x(\lambda) = \sum_{j=1}^{m} \ell b_j \exp(\lambda\omega_j) \quad \text{where} \quad \omega_j = \sum_{i=o}^{\infty} c_{i,j}\, \ell^{\frac{i-p}{q}}$$

while the approximate one is ([1]):

$$(7) \qquad \tilde{x}(\lambda) = \sum_{s=1}^{m} \ell b_j \exp(\lambda\tilde{\omega}_j) \quad \text{where} \quad \tilde{\omega}_j = \sum_{i=o}^{i_o} c_{i,j}\, \ell^{\frac{i-p}{q}} \ ;$$

We shall suppose that $\tilde{b}_j \in C$, where $\{\tilde{b}_j\} = \ell b_j$.

In (6) and (7) an exponential operator $e^{\lambda\omega}$ appears. The conditions for the existence of that operator, together with its character, are given in [1]. Now, in this paper we suppose that $p/q < -1$, so then $(e^{\lambda\omega}-I)$ belongs to $\mathcal{C}$. $\mathcal{C}$ is the ring of continuous complex valued functions defined over $[0,\infty)$.

After a change of variables $t = \tau + T_1$, equation (2) becomes:

$$(8) \qquad \sum_{\mu=o}^{m} \sum_{\nu=o}^{1} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda,\tau+T_1)}{\partial\lambda^{\mu}\,\partial\tau^{\nu}} = 0 \,.$$

Let us denote $x(\lambda,\tau+T_1) = y(\lambda,\tau)$ and observing that:

$$(9) \qquad e^{-T_1 s}\{y(\lambda,\tau)\} = \left\{ \begin{array}{ll} y(\lambda,\tau-T_1), & \tau > T_1 \\ 0, & \tau < T_1 \end{array} \right\} \equiv X(\lambda,\tau)$$

we obtain:

$$\{y(\lambda,\tau)\} = \{x(\lambda,\tau+T_1)\} = e^{T_1 s}\{X(\lambda,\tau)\} = e^{T_1 s} X(\lambda)$$

and

$$\left\{\frac{\partial x(\lambda,\tau+T_1)}{\partial\tau}\right\} = s\{x(\lambda,\tau+T_1)\} - x(\lambda,T_1)I = e^{T_1 s} sX(\lambda) - x(\lambda,T_1)I$$

Equation (8) corresponds in field M to the equation:

$$(10) \qquad e^{T_1 s} \sum_{\mu=o}^{m} \sum_{\nu=o}^{1} \alpha_{\mu,\nu} s^{\nu} X^{(\mu)}(\lambda) = F_1(\lambda)$$

where

$$(11) \qquad F_1(\lambda) = \sum_{\mu=o}^{m} \alpha_{\mu,1} \frac{\partial^{\mu} x(\lambda,t)}{\partial\lambda^{\mu}} \Big|_{t=T_1} .$$

The solution of equation (10) with conditions in M:

$$X(0) = X'(0) = \ldots = X^{(m-2)}(0) = 0$$

$$(12) \qquad X^{(m-1)}(0) = \ell$$

(which follows from (4)) can be formed in the following way ([3]):

$$(13) \qquad X(\lambda) = e^{-T_1 s} \int_0^{\lambda} F(\chi) X_h(\lambda-\chi)\,d\chi,$$

where $F(\chi) = \dfrac{F_1(\chi)}{\ell(\alpha_{m,o}+\alpha_{m,1}s)} = \dfrac{F_1(\chi)}{C}$

and $X_h(\lambda)$ is the solution of the homogeneous part of equation (10), In fact, it has the same form as the solution $x(\lambda)$ of equation (5) given by (6), so we have:

$$X(\lambda) = x(\lambda) + e^{-T_1 s}\int_0^{\lambda} F(\chi)x(\lambda-\chi)d\chi . \tag{14}$$

Integrals in (13) and (14) exist because $F(\chi)x(\lambda-\chi)$ is a continuous function.

Let us observe that the solution of equation (2) $x(\lambda,t)$ can be written as $X(\lambda,\tau)$ for $t=\tau+T_1$ and $t\in[T_1,T]$.

Now, let us replace the exact solution $x(\lambda)$ with an approximate one $\tilde{x}(\lambda)$ given by (7); so instead of equation (10) we have:

$$e^{T_1 s}\sum_{\mu=o}^{m}\sum_{\nu=o}^{1}\alpha_{\mu,\nu}s^{\nu}\tilde{X}^{(\mu)}(\lambda) = \tilde{F}_1(\lambda) \tag{15}$$

where

$$\tilde{F}_1(\lambda) = \sum_{\mu=o}^{m}\alpha_{\mu,1}\frac{\partial^{\mu}\tilde{x}(\lambda,t)}{\partial\lambda^{\mu}}\Big|_{t=T_1} . \tag{16}$$

The solution of equation (15) with conditions (12) has the from

$$\tilde{X}_1(\lambda) = X_h(\lambda) + e^{-T_1 s}\int_0^{\lambda}\tilde{F}(\chi)X_h(\lambda-\chi)d\chi,\quad \tilde{F}(\chi) = \frac{\tilde{F}_1(\chi)}{C} \tag{17}$$

and the approximate solution of (15) is

$$\tilde{X}(\lambda) = \tilde{x}(\lambda) + \tilde{X}_p(\lambda) = \tilde{x}(\lambda) + e^{-T_1 s}\int_0^{\lambda}\tilde{F}(\chi)\tilde{x}(\lambda-\chi)d\chi \tag{18}$$

where $\tilde{x}(\lambda)$ is the approximate solution of the homogeneous part of (15). In fact it has the same form as the approximate solution of equation (5) given by (7).

Let us remark that equation (10) and (15) differ only in their right-hand sides. In the next section we shall prove that the solution of equation (10) depends continuously on the right-hand side. This fact enables us to use $\tilde{X}(\lambda,\tau)$, (where $\{\tilde{X}(\lambda,\tau)\} = \tilde{X}(\lambda)$, $\tilde{X}(\lambda)$ is given by (18)) as the approximate solution of equation (2) on the interval $[T_1,T]$ and to find the error of approximation on $[T_1,T]$ as the difference between $X(\lambda)$ and $\tilde{X}(\lambda)$.

MEASURE OF APPROXIMATION

In this section we use the notion of an absolute value (module) of certain operators from $\mathcal{M}$. If an operator a is defined by a function $a(t), t \geq 0$ from $\mathcal{L}$, then its absolute value $|a| = |\{a(t)\}| = \{|a(t)|\}$ is again a function from $\mathcal{L}$ ([1]. [3]). $\mathcal{L}$ is the ring of locally integrable functions over $[0,\infty)$.

If $g(\lambda,\chi)$ is a continuous operator function, there exists $q \in \mathcal{M}$ such that $\{\tilde{g}_1(\lambda,\chi,t)\} = qg(\lambda,\chi)$ and $\tilde{g}_1(\lambda,\chi,t)$ is a continuous function, then:

$$\left| \int_0^\lambda qg(\lambda,\chi)\,d\chi \right| \leq_T \ell \int_0^\lambda G(\lambda,\chi)\,d\chi$$

where $G(\lambda,\chi) = \max\limits_{0 \leq t \leq T} |\tilde{g}_1(\lambda,\chi,t)|$.

The other properties of the absolute value which we are going to use are given in [1].

First, let us estimate the difference between $F(\lambda)$ given by (11) and $\tilde{F}(\lambda)$ given by (16). For that purpose we note that in the field $\mathcal{M}$ the μ-th derivative of $x(\lambda)$ given by (6) has the following form:

$$x^{(\mu)}(\lambda) = \sum_{j=1}^{m} \ell b_j \omega_j^{\mu} \exp(\lambda\omega_j). \tag{19}$$

In the same manner we take:

$$\tilde{x}^{(\mu)}(\lambda) = \sum_{j=1}^{m} \ell b_j \tilde{\omega}_j^{\mu} \exp(\lambda\tilde{\omega}_j). \tag{20}$$

We shall also need the following inequalities (for $(p/q) < -1$), [1], [2]:

$$\left| \sum_{i=0}^{i_0} c_{i,j} \ell^{(i-p)/q} \right| \leq_T \ell \sum_{i=0}^{i_0} |c_{i,j}| \frac{T^{(i-p)/q-1}}{\Gamma((i-p)/q)} = \nu_j(T)\ell$$

(21)

$$\left| \left(\sum_{i=i_0+1}^{\infty} c_{i,j} \ell^{(i-p)/q} \right)^k \right| \leq \left(\ell^{(i_0-p+1-q)/q} \rho_j^{i_0+1} \sum_{i=0}^{\infty} M\rho_j^i \ell^{i/q+1} \right)^k \leq_T$$

$$\leq \left(\frac{T^{\frac{i_o+1-p-q}{q}}\cdot\rho^{i_o+1}}{\Gamma\left(\frac{i_o+1-p}{q}\right)}\; M\,\ell \sum_{i=o}^{\infty} \rho_j \frac{T^{i/q}}{\Gamma((i/q)+1)}\right)^k \leq_T \gamma_j^k(T)\,\frac{T^{k-1}}{(k-1)!}\,\ell\,,$$

$$|\,\ell \exp(\lambda \sum_{i=o}^{i_o} c_{i,j}\ell^{(i-p)/q})\,| \leq \ell \sum_{k=o}^{\infty} \frac{\lambda^k}{k!}\,|\sum_{i=o}^{i_o} c_{i,j}\ell^{(i-p)/q}|^k \leq_T$$

$$\leq_T \ell \sum_{k=o}^{\infty} \frac{\lambda^k}{k!}\,\nu_j^k(T)\,\frac{T^k}{k!} = N_j(\lambda,T)\,\ell\,.$$

$$|\exp(\lambda \sum_{i=i_o+1}^{\infty} c_{i,j}\ell^{(i-p)/q}) - I\,| \leq_T \lambda\gamma_j(T) \sum_{k=o}^{\infty} \frac{\lambda^k}{k!}\,\gamma_j^k(T)\,\frac{T^k}{k!}\,\ell =$$

$$= {}_T G_j(\lambda,T)\,\ell\,.$$

Since the solution $x(\lambda,t)$ and the approximate one $\tilde{x}(\lambda,t)$ together with their partial derivatives by λ up to the order m are continuous on the set $\{(\lambda,t)\,|\ 0 \leq \lambda \leq \lambda_o,\ 0 \leq t \leq T\}$, there exist numbers $R_\mu(\lambda,T)$ (which we shall determine later in Lemma 1.) so that:

$$(22)\qquad |x^{(\mu)}(\lambda) - \tilde{x}^{(\mu)}(\lambda)| \leq_T R_\mu(\lambda,T)\,\ell\quad,\quad \mu=0,\dots,m\,.$$

This implies:

$$\left|\ \frac{\partial^\mu x(\lambda,t)}{\partial\lambda^\mu}\,\right|_{t=T} - \left.\frac{\partial^\mu \tilde{x}(\lambda,t)}{\partial\lambda^\mu}\,\right|_{t=T} \leq R_\mu(\lambda,T)\,.$$

Using (11), (16) and (22) we have:

$$(23)\qquad |F_1(\lambda)-\tilde{F}_1(\lambda)| \leq \sum_{\mu=o}^{m} |\alpha_{\mu,1}|\,R_\mu(\lambda,T) = \varepsilon(\lambda,T)$$

From |2| and notation (21) follows (if $(p/q) < -1$):

$$(24)\qquad |x(\lambda)-\tilde{x}(\lambda)| \leq_T \ell \sum_{j=1}^{m} |b_j|\,N_j(\lambda,T)\,G_j(\lambda,T)\,T = {}_T R_o(\lambda,T)\,\ell$$

LEMMA 1. *The error of approximation of the μ-th derivative is:*

$$(25)\qquad \begin{aligned} |x^{(\mu)}(\lambda)-\tilde{x}^{(\mu)}(\lambda)| \leq_T \sum_{j=1}^{m} |b_j|\,N_j(\lambda,T)\,\Big(&\nu_j^\mu(T)\,G_j(\lambda,T)\,\frac{T^\mu}{\mu!}\,\ell\,+ \\ &+ \sum_{r=1}^{\mu} \binom{\mu}{r}\,\nu_j^{\mu-r}(T)\,\gamma_j^r(T)\,\Big(\sum_{k=o}^{\infty} \frac{\lambda^k}{k!}\,\gamma_j^k(T)\,\frac{T^k}{k!}\Big)\,\ell^{\mu-1}\Big)\,\ell \end{aligned}$$

P r o o f. In view of (19) and (20) we can write:

$$|x^{(\mu)}(\lambda)-\tilde{x}^{(\mu)}(\lambda)| = \sum_{j=1}^{m} |b_j||\omega_j^{\mu}\ell\exp(\lambda\omega_j)-\tilde{\omega}_j^{\mu}\ell\exp(\lambda\tilde{\omega}_j)| \le$$

$$\le \sum_{j=1}^{m}|b_j||\ell\exp(\lambda\tilde{\omega}_j)||(\tilde{\omega}_j + \sum_{i=i_o+1}^{\infty} c_{i,j}\ell^{(i-p)/q})^{\mu}\exp(\lambda\sum_{i=i_o+1}^{\infty} c_{i,j}$$

$$\cdot\ \ell^{(i-p)/q}) - \tilde{\omega}_j^{\mu}| \le \sum_{j=1}^{m} |b_j||\ell\exp(\lambda\tilde{\omega}_j)|(|\tilde{\omega}_j^{\mu}|\ \cdot$$

$$\cdot\ |\exp(\lambda\sum_{i=i_o+1}^{\infty} c_{i,j}\ell^{(i-p)/q}) - I| +$$

$$+ |(\sum_{r=1}^{\mu}\binom{\mu}{r}\tilde{\omega}_j^{\mu-r}(\sum_{i=i_o+1}^{\infty} c_{i,j}\ell^{(i-p)/q})^{r})\exp(\lambda\sum_{i=i_o+1}^{\infty} c_{i,j}\ell^{(i-p)/q})|$$

$$\le {}_T\sum_{j=1}^{m} |b_j|N_j(\lambda,T)(\nu_j^{\mu}(T)G_j(\lambda,T)\frac{T^{\mu}}{\mu!}\ell +$$

$$+ (\sum_{r=1}^{\mu}\binom{\mu}{r}\nu_j^{\mu-r}(T)\gamma_j^{r}(T))(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\gamma_j^k(T)\frac{T^k}{k!})\ell^{\mu-1})\ell \equiv {}_TR_{\mu}(\lambda,T)\ell\ .$$

LEMMA 2. *The solution* $X(\lambda)$ *given by (13) of equation (10) with conditions (12) depends continuously on* $F(\lambda)$.

P r o o f. If $\tilde{X}_1(\lambda)$, given by (17), is the solution of equation (15) with conditions (12) (equations (10) and (15) differ only in their right-hand sides) then using relations (21) and (23) we have:

$$|X(\lambda)-\tilde{X}_1(\lambda)| = |\int_0^{\lambda} (F(\chi)-\tilde{F}(\chi))e^{-T_1 s}x(\lambda-\chi)d\chi| \le$$

$$\le {}_T\lambda\varepsilon(\lambda,T_1)\overline{N}_j(\lambda,T-T_1)(G_j(\lambda,T-T_1)\ell + \ell).$$

$$\text{where}\quad |\frac{\ell}{C}\exp(\lambda\sum_{i=0}^{i_o} C_{i,j}\ell^{\frac{\ell-p}{2}})| \le {}_T\overline{N}_j(\lambda,T)$$

Lemma 2 enables us to use $\tilde{X}(\lambda)$ from relation (18) as the approximate solution in the field M of equation (10).

PROPOSITION. *If $X(\lambda)$ is the solution of equation (10) given by (13) and $\tilde{X}(\lambda)$ is the approximate one of equation (15) given by (18) then the error of approximation is:*

$$(26)\quad |X(\lambda)-\tilde{X}(\lambda)| \leq_T \sum_{j=1}^{m} |b_j|\lambda\bar{N}_j(\lambda,T-T_1)\ell(\varepsilon(\lambda,T_1) +$$

$$+ G_1(\lambda,T-T_1)(T-T_1)(\varepsilon(\lambda,T_1)+\tilde{F}(\lambda)))+R_o(\lambda,T-T_1)\ell$$

P r o o f. Using (13) and (18) we can write:

$$|X_p(\lambda)-\tilde{X}_p(\lambda)| = |\int_0^\lambda e^{-T_1 s}(F(\chi)x(\lambda-\chi)-\tilde{F}(\chi)\tilde{x}(\lambda-\chi))d\chi| =$$

$$= |\int_0^\lambda e^{-T_1 s}(F(\chi)x(\lambda-\chi)-\tilde{F}(\chi)x(\lambda-\chi)+\tilde{F}(\chi)x(\lambda-\chi)-\tilde{F}(\chi)\tilde{x}(\lambda-\chi))d\chi|$$

$$\leq \int_0^\lambda |(F(\chi)-\tilde{F}(\chi))e^{-T_1 s}\tilde{x}(\lambda-\chi)|d\chi + \int_0^\lambda |(F(\chi)-\tilde{F}(\chi))e^{-T_1 s}(x(\lambda-\chi) -$$

$$- \tilde{x}(\lambda-\chi))|d\chi + \int_0^\lambda |\tilde{F}(\chi)e^{-T_1 s}(x(\lambda-\chi)-\tilde{x}(\lambda-\chi))|\, d\chi .$$

In view of (21),(23),(24) and (25) we have:

$$|X(\lambda)-\tilde{X}(\lambda)| \leq_T \sum_{j=1}^{m} |b_j|\lambda\tilde{N}_j(\lambda,T-T_1)(\varepsilon(\lambda,T_1)\ell +$$

$$+ G_j(\lambda,T-T_1)(T-T_1)\ell(\varepsilon(\lambda,T_1)+\tilde{F}(\lambda))) + R_o(\lambda,T-T_1)\ell .$$

EXAMPLE. The following example will show the advantage of approximation in two steps compared to the older method from [1], [2].

Let us observe the partial differential equation:

$$(27)\qquad \frac{\partial^2 x(\lambda,t)}{\partial\lambda\partial t} - \frac{\partial x(\lambda,t)}{\partial\lambda} - x(\lambda,t) = 0$$

with conditions:

$$(28)\qquad \frac{\partial x(\lambda,0)}{\partial\lambda} = 0,\ \lambda > 0\ ;\quad x(0,t) = 1,\quad t > 0 .$$

In the field M, equation

$$(29)\qquad (s-1)x'(\lambda) - x(\lambda) = 0$$

corresponds to the equation (27) with (28). The solution of equation (29) is:

$$x(\lambda) = \ell\exp(\lambda\omega), \quad \text{where } \omega = \sum_{i=0}^{\infty} \ell^{i+1} \quad \text{and} \quad x'(\lambda) = \ell\omega\exp(\lambda\omega)$$

while the approximate solution of equation (29) is:

$$\tilde{x}(\lambda) = \ell\exp(\lambda\tilde{\omega}), \quad \text{where } \tilde{\omega} = \sum_{i=0}^{i_0} \ell^{i+1} \quad \text{and} \quad \tilde{x}'(\lambda) = \ell\tilde{\omega}\exp(\lambda\tilde{\omega}).$$

After a change of variables $t = \tau + T_1$ equation (27) becomes:

$$(30)\qquad \frac{\partial^2 x(\lambda,\tau+T_1)}{\partial\lambda\ \partial\tau} - \frac{\partial x(\lambda,\tau+T_1)}{\partial\lambda} - x(\lambda,\tau+T_1) = 0$$

with conditions:

$$\frac{\partial x(\lambda,\tau+T_1)}{\partial\lambda}\Big|_{\tau=0} = \phi(\lambda), \qquad x(0,\tau) = 1 .$$

In the field M, equation

$$(31)\qquad e^{T_1 s}(s-1)X'(\lambda) - X(\lambda) = F_1(\lambda)$$

corresponds to equation (30). The solution of equation (31) is:

$$(32)\qquad X(\lambda) = e^{-T_1 s}\int_0^{\lambda} F(X)x(\lambda-X)\,dX + x(\lambda) .$$

If we take $\tilde{\phi}(\lambda) = \dfrac{\partial\tilde{x}(\lambda,\tau+T_1)}{\partial\lambda}\Big|_{\tau=0}$ instead of $\phi(\lambda)$ we get equation:

$$(33)\qquad e^{T_1 s}(s-1)\tilde{X}'(\lambda) - \tilde{X}(\lambda) = \tilde{F}_1(\lambda).$$

From Lemma 2 and [1] the approximate solution of equation (33) given by:

$$(34)\qquad \tilde{X}(\lambda) = e^{-T_1 s}\int_0^{\lambda} \tilde{F}(X)\tilde{x}(\lambda-X)\,dX + \tilde{x}(\lambda)$$

can be observed as the approximate solution of equation (31) and the function $\tilde{X}(\lambda,\tau)$ $(\tilde{X}(\lambda) = \{\tilde{X}(\lambda,\tau)\})$ as the approximate

solution of equation (27) on interval $[T_1,T]$.

Since $m=1$, $b_j=1$, $M=\rho=1$, we can write the entities in (21) without index j. Also, $F_1(\lambda)=\phi(\lambda)$ and $\tilde{F}_1(\lambda)=\tilde{\phi}(\lambda)$ and therefore we can obtain $\varepsilon(\lambda,T)$ using estimation (25) for $\mu=1$. So we have:

$$(35)\quad |x'(\lambda)-\tilde{x}'(\lambda)| \leq_T N(\lambda,T)(\nu(T)G(\lambda,T)T + \\ + \gamma(T)(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\gamma^k(T)\frac{T^k}{k!}))\ell = \varepsilon(\lambda,T)\ell .$$

If the exact solution of equation (31) is given by (32) and the approximate one is given by (34) then the measure of approximation is:

$$|X(\lambda)-\tilde{X}(\lambda)| \leq_T \bar{N}(\lambda,T-T_1)(\varepsilon(\lambda,T_1)+G(\lambda,T-T_1)(T-T_1)(\varepsilon(\lambda,T_1) + \\ + \tilde{\phi}(\lambda))\ell + R_o(\lambda,T-T_1)\ell .$$

In the following table there are two errors of approximation, the first (one step) on interval $[0,T]$ and the other (two steps) on the interval $[T_1,T]$. Let us remark that the error of approximation is smaller when we work in two steps, especially if T is bigger than 1.

T	one step $i_o=5,\lambda=1$	T_1	Two steps $i_o=5,\ \lambda=1$
0,1	$1\cdot10^{-10}$		
0,2	$2,77\cdot10^{-8}$	0,1	$2,\ 1\cdot10^{-9}$
0,4	$6,17\cdot10^{-6}$	0,2	$2,56\cdot10^{-7}$
0,5	$4,08\cdot10^{-5}$	0,1	$1,02\cdot10^{-5}$
0,8	$3,80\cdot10^{-3}$	0,4	$5,75\cdot10^{-5}$
1	$5,73\cdot10^{-2}$	0,5	$4,51\cdot10^{-4}$
1,6	$5,85\cdot10^{3}$	0,8	$1,14\cdot10^{-1}$
2	$1,00\cdot10^{-7}$	1	$5,75\cdot10^{0}$

It is obvious that this method can be applied in the same way in many steps, by dividing the interval $[0,T]$ into n parts which are not neccessarily equal.

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REZIME

PRIBLIŽNO REŠENJE DIFERENCIJALNE JEDNAČINE U KORACIMA

U ovom radu se konstruiše približno rešenje diferencijalne jednačine:

$$\sum_{\mu=o}^{m} \sum_{\nu=o}^{1} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda,t)}{\partial\lambda^{\mu}\, \partial t^{\nu}} = 0$$

sa sledećim uslovima:

$$\frac{\partial^{\mu} x(0,t)}{\partial\lambda^{\mu}} = 0 \qquad \lambda > 0,\ \mu = 0,\dots,m$$

$$\frac{\partial^{\mu} x(0,t)}{\partial\lambda^{\mu}} = 0 \qquad t > 0,\ \mu = 0,\dots,m-2$$

$$\frac{\partial^{m-1} x(0,t)}{\partial\lambda^{m-1}} = 1 \qquad t > 0$$

na intervalu $[0,T]$ u koracima. Znajući približno rešenje na intervalu $[0,T_1]$, $T_1 < T$, konstruiše se približno rešenje na intervalu $[T_1,T]$ i ocenjuje greška.

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FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

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ABSTRACT

Most fixed point theorems for Probabilistic Metric spaces (PM-spaces) have been proved for the same subclass of PM-spaces. It is shown that this subclass is metrizable. Furthermore, the compatible metric d is related to the distribution functions by

$$d(x,y) < t \text{ if and only if } F_{x,y}(t) > 1-t \text{ .}$$

This allows an exact translation of the contraction condition, as well as other conditions studied in metric spaces, to PM-spaces. Thus, theorems follow immediately from corresponding theorems for metric spaces.

1. INTRODUCTION

A real-valued function defined on the set of real numbers is a <u>distribution function</u> if it is nondecreasing, left continuous and $\inf f = 0$, $\sup f = 1$. H denotes the distribution function defined by $H(x) = 0$ if $x \leq 0$, and $H(x) = 1$ for $x > 0$.

DEFINITION 1.1. *Let* X *be a set and* F *be a function on* $X \times X$ *such that* $F(x,y) = F_{xy}$ *is a distribution function. Consider the following conditions:*

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I. $F_{x,y}(0) = 0$ *for all* x,y *in* X.

II. $F_{x,y} = H$ *if and only if* $x = y$.

III. $F_{x,y} = F_{y,x}$.

IV. *If* $F_{x,y}(\varepsilon) = 1$ *and* $F_{y,z}(\delta) = 1$, *then* $F_{x,z}(\varepsilon+\delta) = 1$.

IV_m. $F_{x,z}(\varepsilon+\delta) \geq T(F_{x,y}(\varepsilon), F_{y,z}(\delta))$.

If $\mathcal{F}$ satisfies conditions I and II then it is called a pre-probabilistic metric structure (PPM-structure) on X and the pair $(X,\mathcal{F})$ is called a pre-probabilistic metric space (PPM-space). An $\mathcal{F}$ satisfying condition III is said to be symmetric. A symmetric PPM-structure $\mathcal{F}$ satisfying IV is a probabilistic metric structure (PM-structure) and the pair $(X,\mathcal{F})$ is a probabilistic metric space (PM-space).

DEFINITION 1.2. *A Menger space is a* PM-*space that satisfies* IV_m, *where* T *is a* 2-*place function on the unit square satisfying:*

1. $T(0,0) = 0$, $T(a,1) = a$,
2. $T(a,b) = T(b,a)$,
3. *if* $a \leq c$, $b \leq d$, *then* $T(a,b) \leq T(c,d)$,
4. $T(T(a,b),c) = T(a,T(b,c))$.

T *is called a* t-*norm.*

Let $(X,\mathcal{F})$ be a PPM-space. For $\varepsilon, \lambda > 0$ and $x \in X$, let $N_x(\varepsilon,\lambda) = \{y : F_{x,y}(\varepsilon) > 1-\lambda\}$.

A T_1 topology $\tau(\mathcal{F})$ on X is obtained as follows: $U \in \tau(\mathcal{F})$ if for each $x \in U$, there exists $\varepsilon > 0$ such that $N_x(\varepsilon,\varepsilon) \subset U$. The study of fixed point theory in probabilistic metric spaces (PM-spaces) was started by Sehgal and Bharucha-Reid [10]. The following definition and theorem appeared in their paper.

DEFINITION 1.3. *A mapping* f *of a PM-space* (X,F) *into itself is a contraction if there exists* k, *with* $0<k<1$, *such that for each* $x,y \in X$,

$$F_{fx,fy}(kt) \geq F_{x,y}(t) \text{ for all } t>0.$$

THEOREM 1.1. *Let* (X,F,T) *be a complete Menger space where* $T(a,b)=\min\{a,b\}$. *If* f *is any contraction, there exists a unique* $p \in X$ *such that* $f(p)=p$. *Moreover,* $\lim f^n(q)=p$ *for each* $q \in X$.

A little thought convinces oneself that this is a reasonable definition in this new setting. Also, if f is a contraction ($d(fx,fy) \leq k\, d(x,y)$) on a complete metric space (X,d), and one makes it into a PM-space in the natural way; that is,

$$F_{x,y}(t) = H(t-d(x,y)),$$

then $F_{fx,fy}(kt) \geq F_{x,y}(t)$. In [1], it was shown that the weaker condition,

$$F_{fx,fy}(kt) \geq F_{x,y}(t) \quad \text{whenever} \quad F_{x,y}(t) > 1-t\ ,$$

is sufficient to obtain the above theorem. As originally given, the theorem required T to be continuous and satisfy $T(x,x) \geq x$. It is easy to see that this forces $T(a,b)=\min\{a,b\}$.

2. BASIC THEOREMS

The following condition is another reasonable generalization of a contraction to PM-spaces.

(c) For $t>0$, $F_{fx,fy}(kt) > 1-kt$ whenever $F_{x,y}(t) > 1-t$.

REMARK 1. If the metric space (X,d) is made into a PM-space as indicated above; that is, $F_{xy}(t)=H(t-d(x,y))$, then if $d(fx,fy) \leq k\, d(x,y)$, for $0<k\leq 1$, we have condition (c).

P r o o f. $F_{fx,fy}(kt) = H(kt - d(fx,fy)) \geq H(kt-kd(x,y)) = H(t - d(x,y)) = F_{x,y}(t)$. Now $F_{x,y}(t) = H(t-d(x,y)) > 1-t$ if and only if $F_{x,y}(t) = 1$ if and only if $F_{x,y}(t) > 1-kt$. Condition (c) follows.

We now show that for each PM-space in a class larger than the one described in Theorem 1.1, there exists a compatible metric d such that

$$d(fx,fy) \leq k\, d(x,y) \quad \text{iff (c) holds.}$$

Then, using condition (c) as our definition of a contraction, we have Banach's theorem for PM-spaces as a consequence of Banach's theorem for metric spaces. Actually, a nicer result is obtained that allows you to translate many other fixed point theorems for metric spaces to PM-spaces. The result that makes this possible is:

$$d(x,y) < t \quad \text{iff} \quad F_{x,y}(t) > 1-t .$$

THEOREM 2.1. *Let* (X,F) *be a symmetric* PPM-*space such that*

$$F_{x,z}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\} .$$

Let $d(x,y) = \begin{cases} \sup\{\varepsilon : y \notin N_x(\varepsilon,\varepsilon),\ 0 < \varepsilon < 1\}, \\ 0 \text{ if } \ y \in N_x(\varepsilon,\varepsilon) \text{ for all } \varepsilon > 0 . \end{cases}$

Then

(1) $d(x,y) < t$ *if and only if* $F_{x,y}(t) > 1-t$.

(2) d *is a compatible metric for* t(F).

(3) *If* $f:X \to X$ *and* $0 < k \leq 1$,

(c) *holds if and only if* $d(fx,fy) \leq k\, d(x,y)$.

(4) (X,F) *is complete if and only if* (X,d) *is complete.*

P r o o f. Observe that if $t < r$, $N_x(t,t) \subset N_x(r,r)$. Also, $\bigcap\{N_x(\varepsilon,\varepsilon) : 0 < \varepsilon < 1\} = \{x\}$. For, if $x \neq y$, $F_{x,y} \neq H$. Thus

there exists $\varepsilon > 0$ such that $F_{x,y}(\varepsilon) = \delta$ where $0 < \delta < 1$. Set $\delta = 1-\delta_1$ and let $\varepsilon_1 = \min\{\varepsilon, \delta_1\}$. Then $F_{x,y}(\varepsilon_1) \leq F_{x,y}(\varepsilon) = \delta = 1-\delta_1 \leq 1-\varepsilon_1$ gives $y \notin N_x(\varepsilon_1, \varepsilon_1)$.

(1) If $1 < t$, $d(x,y) \leq 1 < t$ and also $F_{x,y}(t) \geq 0 > 1-t$. Suppose $d(x,y) < t \leq 1$. Choose δ such that $d(x,y) < \delta < t \leq 1$. Then $y \in N_x(\delta,\delta)$ and $F_{x,y}(t) \geq F_{x,y}(\delta) > 1-\delta > 1-t$. For, if we assume $y \notin N_x(\delta,\delta)$, then $d(x,y) = \sup\{\ \} \geq \delta$, a contradiction. Conversely, suppose $F_{x,y}(t) > 1-t$ where $0 < t \leq 1$. Then $y \in N_x(t,t)$. If $y \notin N_x(\varepsilon,\varepsilon)$ for all $\varepsilon < t$, $F_{x,y}(t) = \lim_{\varepsilon \to t^-} F_{x,y}(\varepsilon) \leq \lim_{\varepsilon \to t^-} (1-\varepsilon) = 1-t$, a contradiction. Thus there exists $0 < \varepsilon < t$ such that $y \in N_x(\varepsilon,\varepsilon)$. Hence $d(x,y) \leq \varepsilon < t$.

(2) If d satisfies the triangular inequality, it is a metric. Also, (1) shows it is compatible with $t(F)$. We observe that $d(x,y) < \varepsilon_1$ and $d(y,z) < \varepsilon_2$ implies that $d(x,z) < \varepsilon_1 + \varepsilon_2$. For, suppose

$$F_{x,y}(\varepsilon_1) > 1-\varepsilon_1 \quad \text{and} \quad F_{y,z}(\varepsilon_2) > 1-\varepsilon_2 .$$

If $F_{x,y}(\varepsilon_1)$ is the minimum,

$$F_{x,z}(\varepsilon_1+\varepsilon_2) \geq \min\{F_{x,y}(\varepsilon_1),\ F_{y,z}(\varepsilon_2)\} > 1-\varepsilon_1 > 1-(\varepsilon_1+\varepsilon_2)$$

gives $d(x,z) < \varepsilon_1+\varepsilon_2$. The triangular inequality follows.

(3) Suppose $d(fx,fy) \leq k\, d(x,y)$ and $F_{x,y}(t) > 1-t$. Then $d(x,y) < t$ and $d(fx,fy) < kt$. Thus $F_{fx,fy}(kt) > 1-kt$. If (c) holds, let $\varepsilon > 0$ be given. Set $t = d(x,y) + \varepsilon$. $d(x,y) = t-\varepsilon < t$ gives

$$F_{x,y}(t) > 1-t, \text{ and } F_{fx,fy}(kt) > 1-kt$$

follows from (c). Thus $d(fx,fy) < kt = k(d(x,y)+\varepsilon) = kd(x,y)+k\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d(fx,fy) \leq k\, d(x,y)$.

REMARK 2. Assuming the conditions in Theorem 1.1, we have

$$F_{x,z}(r+s) \geq T(F_{x,y}(r), F_{y,z}(s)) = \min\{F_{x,y}(r), F_{y,z}(s)\} ,$$

the inequality in Theorem 2.1. Also, the inequality in Theorem 2.1 does not require the existence of a t-norm. Condition (c) and the earlier definition of contraction seem to be independent for $0 < k < 1$.

COROLLARY. *Let* (X,F) *be a complete symmetric* PPM-*space such that*

$$F_{x,y}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\} .$$

Suppose $f: X \to X$ *satisfies (c). Then* f *has a unique fixed point* p. *Also, if* $x \in X$ *and* $x_n = f^n(x)$, *then*

(1) $p = \lim x_n$, *and*

(2) *for* $t \geq \frac{k^{n-1}}{1-k} d(x,fx) = \alpha_n$,

$$1 - F_{x_n,p}(t) \leq \frac{k^{n-1}}{1-k} d(x,f(x)) .$$

P r o o f. The theorem gives a compatible metric d such that $d(fx,fy) \leq k\, d(x,y)$. From Banach´s fixed point theorem, f has a unique fixed point p satisfying (1). Also,

$$d(x_n,p) \leq \frac{k^n}{1-k} d(x,fx) < \frac{k^{n-1}}{1-k} d(x,fx) = \alpha_n .$$

From (1) of the Theorem,

$$F_{x_n,p}(\alpha_n) > 1 - \alpha_n .$$

For $t \geq \alpha_n$,

$$F_{x_n,p}(t) \geq F_{x_n,p}(\alpha_n) > 1 - \alpha_n .$$

REMARK 3. Note that the error bound is usable. Given $\varepsilon > 0$, choose $0 < \varepsilon_0 < 1$ and x such that $d(x,fx) < \varepsilon_0$; that is, $F_{x,fx}(\varepsilon_0) > 1 - \varepsilon_0$. For $t \geq \beta = \frac{\varepsilon_0}{1-k} > \alpha_n$,

$$1 - F_{x_n,p}(t) \leq \frac{k^{n-1}}{1-k} d(x,fx) < \frac{k^{n-1}}{1-k} \varepsilon_0 .$$

If $\frac{k^{N-1}}{1-k}\,\varepsilon_0 < \varepsilon$, then $1 - F_{x_n,p}(t) < \varepsilon$ for all $n \geq N$ all $t \geq \beta$.

We next consider how to translate other contractive type conditions for metric spaces to PM-spaces.

LEMMA. *Let* (X,F) *and* d *be as in Theorem 2.1, and* $0 < k \leq 1$. *Let* $R = R(x,y)$ *be a function such that* $d(x,y) \leq R$.

(C*) $F_{fx,fy}(kt) > 1-kt$ *whenever* $F_{x,y}(t) > 1-t$ *and* $t > R$.

Then (C) holds if and only if* $d(fx,fy) \leq kR$.

P r o o f. The proof given for (3) of Theorem 2.1 will work here.

The numbering of the various contractive type conditions are those of Rhoades [9]. Conditions (1),(2) and (3) of [9] have obvious translations using Theorem 2.1. The Lemma can be used on other conditions. We illustrate this with the condition

(24): For $0 \leq k < 1$,

$$d(fx,fy) \leq k \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}.$$

The translation is (C*) of the lemma with $R = \text{Max}\{---\}$. There is a difficulty with this translation since (C*) involves $R = R(x,y)$. Another approach is possible. We translate a condition that gives a common generalization of many of the conditions in [9]. The following theorem was proved by Hicks and Rhoades in [4].

THEOREM 2.2. *Let* (X,d) *be a complete metric space and* $0 \leq k < 1$. *Suppose* f *is a self map of* X, *and there exists an* x *such that*

(A) $d(fy, f^2y) \leq k\, d(y,fy)$

for every $y \in O(x,\infty) = \{x, f(x), f^2(x), \ldots\}$. *Then:*

(i) $\lim f^n x = q$ *exists*.

(ii) $d(f^n x, q) \leq \frac{k^n}{1-k}\, d(x,fx)$.

(iii) *If* f *is continuous at* q, $fq = q$

It was pointed out in [9], that conditions (1),(4), (5),(7),(9),(11),(18) and (19) each imply (21) and (21) is equivalent to (21′).

(21′) For $0 \le k < 1$,

$$d(fx,fy) \le k \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}.$$

It was noted in [4], that (21′) implies (A) for all $y \in X$, and for $0 \le k \le \frac{1}{2}$, (24) implies (A). The following general theorem follows from Theorem 2.1 and 2.2.

THEOREM 2.3. *Let* (X,F) *be as in Theorem 2.1 and* f *a self map of* X. *Suppose there exists an* x *such that*

(A′) *for* $t > 0$, $F_{fy,f^2y}(kt) > 1-kt$ *whenever*

$$F_{x,fy}(t) > 1-t \text{ and } y \in O(x,\infty).$$

Then:

(i) $\lim f^n x = q$ *exists.*

(ii) *If* f *is continuous at* q, $fq = q$.

(iii) *For* $t \ge \frac{k^{n-1}}{1-k} f(x,Tx)$, *we have*

$$1 - F_{x_n,p}(t) \le \frac{k^{n-1}}{1-k} d(x,fx).$$

Thus, for condinuous f, (A′) is more general than the translation of (21′). Also, (A′) refers only to the distribution function. The compatible metric d satisfying $d(x,y) < t$ if and only if $F_{x,y}(t) > 1-t$ allows the translation of many other concepts and theorems from metric spaces to PM-spaces. The following will serve as an illustration.

Let (X,d) be a metric space and let $\varepsilon > 0$. X is ε-chainable if for every $x,y \in X$, there exists $x_0, x_1, \ldots, x_n$ in X such that

$$d(x_i, x_{i+1}) < \varepsilon, \quad i=0,1,\ldots,n-1.$$

For PM-spaces the condition becomes

$$F_{x_i,x_{i+1}}(\varepsilon) > 1-\varepsilon, \quad i=0,1,\ldots,n-1.$$

A mapping f is called an (ε,λ)-local contraction if

$$d(fx,fy) \leq \lambda d(x,y) \text{ whenever } d(x,y) < \varepsilon.$$

This becomes

$$F_{fx,fy}(\lambda t) > 1-\lambda t \text{ whenever } F_{x,y}(\varepsilon) > 1-\varepsilon \text{ and}$$

$$F_{x,y}(\lambda) > 1-\lambda; \text{ that is, whenever}$$

$$F_{x,y}(\alpha) > 1-\alpha \text{ where } \alpha = \min\{\varepsilon,\lambda\}.$$

Edelstein's Theorem [2] for PM-spaces follows.

THEOREM 2.4. *Let* (X,F) *be a complete* ε*-chainable symmetric* PPM*-space such that*

$$F_{x,y}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\}.$$

Suppose $f: X \to X$ is an (ε,λ)-contraction, where $0 < \lambda < 1$. Then f has a unique fixed point p and $\lim_n f^n x = p$ for any x in X.

PROBLEM. Can the condition $F_{x,z}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\}$ in Theorem 2.1 be replaced by some other reasonable (weaker) conditions ?

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REZIME

TEORIJA NEPOKRETNE TAČKE U VEROVATNOSNIM METRIČKIM PROSTORIMA

Dokazane su teoreme o nepokretnoj tački za neke klase verovatnosnih metričkih prostora.

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ON THE PROBABILISTIC INNER MEASURE OF NONCOMPACTNESS

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ABSTRACT

In this paper some properties of the probabilistic inner measure of noncompactness are investigated and a fixed point theorem is proved.

Beginning with Bocsan's work [1], remarkable attention has been paid to probabilistic measures of noncompactness (briefly, probabilistic measures) and their applications to fixed point theory [2-7]. Usually probabilistic measure is assumed to have the properties:

1) $\phi_A(t) = 1$ $(\forall t > 0)$ if and only if A is precompact,

2) $\phi_{\overline{co}A} = \phi_A$,

3) $\phi_{A \cup B} = \min\{\phi_A, \phi_B\}$.

Having been suggested by [8], here we show that for getting fixed point theorems it suffices to assume 1) and that

2') $\phi_{\overline{co}\,A} \geq \phi_A$,

3') $\phi_{A \cup \{x\}} \geq \phi_A$ for each singleton $\{x\}$.

Then, as an example, we give the definition of probabilistic inner measure and establish some of its properties and its relation with the inner measure studied in [8-9].

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1. Let us first recall some definitions. In the sequel we shall use the following notations. R (R^+) stands for the set of all real (non-negative) numbers, 2^X - the family of all nonempty subsets of X, B(X)-the family of all bounded subsets of a locally convex space X, $\overline{co}$ A-the closed convex hull of A.

A function $F:R \to [0,1]$ is called a <u>distribution</u> if it is non-decreasing, left-continuous, $\inf F = 0$, $\sup F = 1$. A <u>random normed space</u> is a pair (X,F) of a given linear space X and a family F of distributions $\{F_x : x \in X\}$ satisfying

a) $F_x(t) = 1$ $(\forall t > 0)$ if and only if $x = \theta$,

b) $F_x(0) = 0$,

c) $F_{cx}(t) = F_x(\frac{t}{|c|})$, $\forall c \neq 0$,

d) $F_{x+y}(t+s) \geq \min\{F_x(t) F_y(s)\}$.

Putting $p_\lambda(x) = \sup\{t : F_x(t) \leq 1-\lambda\}$, $(\lambda \in [0,1])$, we get a semi-norm and (X,p_λ) becomes a Hausdorff locally convex space. In what follows by all the topological notions in (X,F) we mean the corresponding ones in (X,p_λ). Let $\{\phi_A : A \in B(X)\}$ be a family of distributions satisfying 1), 2´), 3´).

DEFINITION 1. *A mapping* $T:X \to 2^X$ *is said to be probabilistic* ϕ*-condensing if* $\phi_{TA} > \phi_A$ *for every* $A \in B(X)$ *which is not precompact.*

Using the method of Reich in [10], we can prove.

THEOREM 2. *Let* (X,F) *be a quasi-complete random normed space , C a nonempty closed convex subset of* $X, T:C \to 2^C$ *an upper semicontinuous probabilistic* ϕ*-condensing mapping having a bounded range. If* $T(x) = \overline{co}T(x)$*, for every x in C then there exists* $x_o \in C$ *such that* $x_o \in Tx_o$*.*

P r o o f. Fixing $z \in C$ we denote $\Phi = \{Y \subset C : z \in Y$, Y is closed, convex and $T(Y) \subset Y\}$. Then $\Phi \neq \emptyset$ (since $C \in \Phi$) and each chain in $(\Phi, \subseteq)$ has a lower bound. So by the Zorn lemma, Φ has a minimal element Z. Denote $V = \overline{co}(T(Z) \cup \{z\})$. Obviously, $V \in \Phi$ and $V \subset Z$, hence $V = Z$. But it follows that Z is bounded and $\phi_{TZ} \leq \phi_Z$, so Z is precompact. Since X is quasi-complete and Z is closed, it must be compact. Being an u.s.c. mapping acting in a compact convex subset Z of a Hausdorff locally convex space X, T has a fixed point by the well-known Ky Fan fixed point theorem.

2. Of course each probabilistic measure with properties 1),2),3) (in particular, the measures α_A and β_A in [1,2]) has properties 1), 2´), 3´). We now present a nontrivial example of probabilistic measure with these properties. Denote $h_{AB}(t) = \sup_{s<t} \inf_{x \in A} \sup_{y \in B} F_{xy}(s)$ and call it the probabilistic nonsymmetric Hausdorff distance between A and B in $B(X)$. Now the probabilistic inner measure of A is defined by $b_A(t) = \sup\{\rho > 0$: there is a finite set $A_f \subset A$ with $h_{AA_f}(t) \geq \rho\}$ for $A \in B(X)$, $t \in R$. Remember that in [3] we defined $\beta_A(t) = \sup\{\rho > 0$: there is a finite set $A_f \subset X$ with $h_{AA_f}(t) \geq \rho\}$, and showed that it coincides with the probabilistic Hausdorff measure introduced by Constantin and Bocsan in [2], where h is replaced by H - the probabilistic Hausdorff distance. Obviously, we have

$$b_A \leq \beta_A \tag{1}$$

It is not difficult to see that b_A is a distribution. Besides, by Proposition 5(8) in [3] (where in the proof A_f was taken in A) we have

$$b_A \geq \alpha_A . \tag{2}$$

From (1) and (2) it follows that b_A has property 1). Further, observe that in the definition of b_A we may replace a finite set by a precompact one, so modifying the proof of Proposition 5(6) in [3] we get property 2´). Property 3´) is also

easy to verify. Obviously, in general b_A is not monotone with respect to A so it need not satisfy 2) and 3).

Moreover, modifying the proof of Proposition 5 in [3] and using condition c) of F_x above we easily get the following further properties of b_A:

4) $b_{cA}(t) = b_A(\frac{t}{|c|})$, $\forall c \neq 0$,

5) $b_{x+A} = b_A$,

6) $b_{A\cup B} \geq \min\{b_A, b_B\}$,

7) $b_{A+B}(t+s) \geq \min\{b_A(t), b_B(s)\}$.

Also, modifying the proof of Proposition 7 in [3] we can see that every probabilistic contraction is probabilistic b-condensing.

3. DEFINITION 3. *A distribution* f *is said to be strict if it is strictly monotone, i.e. for each* $c \in (0,1)$ *the equation* $f(t) = c$ *has at most one solution. Geometrically, it means that the graph of* f *does not contain any horizontal interval outside two lines* $y \equiv 0$ *and* $y \equiv 0$.

In [9] Danes introduced the inner Hausdorff measure as follows:

(3) $\chi(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } A\}$.

We now modify this notion for a locally convex space (X, p_λ) by putting

$$\chi_\lambda(A) = \inf\{\varepsilon > 0: \text{there are } x_1, \dots, x_n \in A \text{ such that } A \subset \cup B_\lambda(x_i, \varepsilon)\}$$

where $B_\lambda(x_i, \varepsilon) = \{x \in X : p_\lambda(x - x_i) < \varepsilon\}$. Obviously this measure has the following properties:

i) $\chi_\lambda(A) = 0$ $(\forall\lambda \in (0,1))$ if and only if A is precompact,

ii) $\chi_\lambda(\overline{co}A) \leq \chi_\lambda(A)$,

iii) $\chi_\lambda(A \cup \{x\}) \leq \chi_\lambda(A)$ for each x in X.

The following result establishes the relation between b_A and χ_λ.

THEOREM 4. *Let* (X,F) *be a random normed space,* b_A *the probabilistic inner measure in* X. *Put*

$$\beta_\lambda(A) = \sup\{t : b_A(t) \le 1-\lambda\} .$$

Then $\chi_\lambda \le \beta_\lambda$. *If* b_A *is strict, we have* $\chi_\lambda = \beta_\lambda$.

Conversely, if χ_λ *is the inner measure which is left-continuous* and *non-increasing in* λ, *then*

$$(4) \qquad \beta_A(t) = 1 - \sup\{\lambda \in (0,1) : \chi_\lambda(A) \ge t\}$$

is a distribution with properties 1), 2′), 3′) *and* $\beta_A \ge b_A$. *Moreover*: b_A *is strict* $\Rightarrow b_A = \beta_A$.

P r o o f. Fixing A and λ we denote $K = \{t : b_A(t) \le 1-\lambda\}$, so $a = \beta_\lambda(A) = \sup K$. First we show that $a \ge \chi_\lambda(A)$. Let $t_o > a$, then $b_A(t_o) > 1-\lambda$. By the definition of b_A we get

$$\sup\{\rho > 0 : \text{there are } x_1,\dots,x_n \in A \text{ with } \sup_{s<t_o} \inf_{x\in A} \max_i F_{xx_i}(s) \ge \rho\} > 1-\lambda .$$

So there are $x_1,\dots,x_n \in A$ such that

$$\sup_{s<t_o} \inf_{x\in A} \max_i F_{xx_i}(s) > 1-\lambda .$$

This implies that there exists an $s_o < t_o$ such that for each $x \in A$ there is an i with $F_{xx_i}(s_o) > 1-\lambda$. This inequality is equivalent to $p_\lambda(x-x_i) < s_o$ (see, for example, [11]). But this implies immediately that $\chi_\lambda(A) \le s_o < t_o$, from this $\chi_\lambda(A) \le$ $\le a = \beta_\lambda(A)$.

Assume now b_A is strict and suppose the contrary that $a > b > c > \chi_\lambda(A)$. Then by (3) there are $x_1,\dots,x_n \in A$ such that for each $x \in A$ there is an i with $p_\lambda(x-x_i) < c$, or equivalently $F_{xx_i}(c) > 1-\lambda$. But it implies

$$h_{A\{x_i\}}(b) = \sup_{s<b} \inf_{x\in A} \max_i F_{xx_i}(s) \ge 1-\lambda .$$

So by the definition of b_A we get $b_A(b) \geq 1-\lambda$. Since b_A is nondecreasing and left-continuous, K is closed, i.e. $a \in K$. But this implies $b_A(a) = b_A(b) = 1-\lambda$, a contradiction to the strictness of b_A and the first part of the theorem is proved.

Now fix A, t and denote $\beta_A(t) = a$. Then we must show that $a \geq b_A(t)$. Suppose the contrary that $a < b_A(t)$. Choose a $\lambda_o \in (0,1)$ so that $0 \leq a \leq b = 1-\lambda_o < b_A(t)$. Then by the definition of b_A, there exist $x_1,\dots,x_n \in X$ such that $\sup_{s<t} \inf_{x\in A} \max_i F_{xx_i}(s) > b$. So there is an $s_o < t$ such that for every $x \in A$ there exists an i with $F_{xx_i}(s_o) > b$ or equivalently, $p_{\lambda_o}(x-x_i) < s_o$. From this $\chi_{\lambda_o}(A) \leq s_o < t$, consequently, $\lambda_o > \sup\{\lambda : \chi_\lambda(A) \geq t\}$, hence $1-\lambda_o = b < \beta_A(t)$, a contradiction. b_A is strict $\Rightarrow b_A = \beta_A$. To prove it, denote $b_\lambda(A) = \sup\{t : b_A(t) \leq 1-\lambda\}$ and recall that $b_A(t) = \sup\{\rho : \exists\{x_i\} \subset A, h_{A\{x_i\}}(t) \geq \rho\}$, $\chi_\lambda(A) = \inf\{\varepsilon : \exists\{x_i\} \subset A, A \subset \cup B_\lambda(x_i,\varepsilon)\}$, $\beta_A(t) = 1 - \sup\{\lambda : \chi_\lambda(A) \geq t\}$. One can prove that $b_A(t) = 1 - \sup\{\lambda : b_\lambda(A) \geq t\}$, so for proving $b_A = \beta_A$ it suffices to show that $\chi_\lambda = b_\lambda$.

First note that $b_\lambda \geq \chi_\lambda$ without any assumption. Indeed denoting $K = \{t : b_A(t) > 1-\lambda\}$, $a = b_\lambda(A)$ we have $a = \inf K$ (here λ and A being fixed). Take $v \in K$, then $b_A(v) > 1-\lambda$ and hence $\exists\{x_i\} \subset A$, $\exists u < v$ such that $\forall x \in A$ $\exists i$ with $F_{xx_i}(u) > 1-\lambda$ but it implies $A \subset \cup B_\lambda(x_i,u)$ and hence $\chi_\lambda(A) \leq v$. So $\chi_\lambda(A) \leq \inf K = a = b_\lambda(A)$.

Now suppose b_A is strict (i.e. $t < s \Rightarrow b_A(t) < b_A(s)$, except for $b_A(t) = b_A(s) = 0$ or 1). We assume the contrary that $a = b_\lambda(A) > a' > \chi_\lambda(A)$. Then $\exists\{x_i\} \subset A$ such that $\forall x \in A$ $\exists i$ with $p_\lambda(x-x_i) < a'$ but it implies $\inf_{x\in A} \max_i F_{xx_i}(a') \geq 1-\lambda$. So $b_A(t) \geq 1-\lambda$ for each $t > a'$. Since b_A is strict, $\inf K = \inf\{t : b_A(t) \geq 1-\lambda\}$. From this, $a' \geq \inf K = a$, a contradiction.

So $a = b_\lambda(A) = \chi_\lambda(A)$.

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REZIME

O VEROVATNOSNOJ UNUTRAŠNJOJ MERI NEKOMPAKTNOSTI

U ovom radu dokazane su neke osobine verovatnosne unutrašnje mere nekompaktnosti.

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ON THE t-NORMS OF THE HADŽIĆ TYPE AND FIXED POINTS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

It is now well known [4] that the Banach principle for probabilistic contractions is valid in complete Menger spaces under a continuous t-norms whose iterations are equicontinuous at x=1. The aim if this note is to give a chatacterization of this class of t-norms and to show that the above mentioned principle can be obtained from the classical. Thus we obtain an improvement of our similar result in [6] where the Min case was considered.

The terminology and the notations are as in [2,10].

DEFINITION. *We shall say that the continuous* t-*norm* T *is an* h-t-*norm if the family* T_m *defined on* $[0,1]$ *by*

$$T_1(x) = x, \quad T_{m+1}(x) = T(T_m(x), x),$$

is equicontinuous at $x = 1$.

Examples of h-t-norms are given in [3,4]. Our following result shows that the h-t-norms have a very simple structure:

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Key words and phrases: Probabilistic metric spaces, t-norms, fixed points.

LEMMA 1. *The following statements are equivalent*

A. T *is an* h-t-*norm;*

B. T *is continuous and* $\forall a > 0$, $\exists b \geq a$ *such that* $T(b,b) = b < 1$.

P r o o f. Suppose that A. holds and let $a > 0$ be given. Then there exists $c > 0$ such that $T_m(x) > a$, $\forall x \geq c$, $\forall m \geq 1$. Since clearly $\{T_m(c)\}$ is nonincreasing, then it is convergent to some limit $b \geq a$. As

$$T_{2m}(c) = T(T_m(c), T_m(c))$$

then $b = T(b,b)$ and we obtain that A. implies B.

Conversely, it is obvious that B. implies A. and the lemma is proved.

REMARK 1. In the proof of A.$\Longrightarrow$B. only the left - equicontinuity at 1 of T_m and the right continuity of T_1 is used. Clearly, the continuity plays no role in B: $\Longrightarrow$A.

REMARK 2. The h-t-norms were considered by O.Hadžić who also constructed an example different from Min [3,4].

The following lemma shows how to construct generalized metrics on a Menger space under an h-t-norm:

LEMMA 2. *Let* T *be an* h-t-*norm. For* $0 < a_1 < a_2 < , \ldots \ldots < a_n \to \infty$, $0 < b_1 < b_2 < \ldots < b_n \to 1$, $T(b_n, b_n) = b_n$ *let us set*

$$F(x) = \begin{cases} 0 & \text{if } x \leq a_1 \\ b_n & \text{if } x \in (a_n, a_{n+1}],\ n=1,2,\ldots \end{cases}$$

Consider a Menger space (S, F, T) *and define*

$$d(p,q) = \inf\{a > 0,\ F_{pq}(ax) \geq F(x), \forall x \in R\}$$

Then (i) d *is a generalized metric on* S*;*

(ii) *If* S *is* F-*complete then* S *is* d-*complete;*

(iii) *The* d-*topology is not weaker than the* F-*topology.*

P r o o f. (i) We prove only the triangle inequality. If $d(p,q) < a' < a$, $d(q,r) < b' < b$ and $x \in (a_n, a_{n+1}]$ then

$$F_{pr}(a'x+b'x) \geq T(F_{pq}(a'x), F(b'x)) \geq$$

$$\geq T_2(F(x)) = T(b_n, b_n) = b_n = F(x) .$$

Therefore $d(p,q) \leq a'+b' < a+b$, and we obtain the triangle inequality.

(ii) and (iii): Let $\{p_n\}$ be a d-Cauchy sequence and fix $a > 0$. By the definition of d, there exists $n_a \geq 1$ such that

$$F_{p_n p_{n+m}}(ax) \geq F(x), \ \forall n \geq n_a, \forall m \geq 1, \ \forall x \in R.$$

If $\varepsilon > 0$ and $\lambda \in (0,1)$ are given, then let $a > 0$ and $z_o \in R$ such that $F(z_o) > 1-\lambda$ and $az_o \leq \varepsilon$.

If $n \geq n_a$, $m \geq 1$ then $F_{p_n p_{n+m}}(\varepsilon) > 1-\lambda$, which shows that p_n is F-Cauchy. If we suppose that S is F-complete then $\{p_n\}$ is F-convergent to some limit p. Therefore

$$F(x) \leq \lim_{m\to\infty} F_{p_n p_{n+m}}(az) = F_{p_n p}(az),$$

for each real z and all $n \geq n_a$, that is $d(p_n,p) \leq a, \forall n \geq n_o$. Thus $\{p_n\}$ is d-convergent and the lemma is proved.

REMARK. For given p_o, q_o in S we can take a_n in the lemma such that $F_{pq}(a_n) \geq b_n$ and the metric d is nontrivial in this case.

The following result was proved in [4]:

THEOREM A. *If* (S,F,T) *is a complete Menger space under an* h-t-*norm then each probabilistic contraction on* S *has a unique fixed point which is the limit of the succesive approximations.*

REMARK 3. As it is well known [7,8,1,2] the Banach contraction principle is a consequence of the above Theorem A. We will prove the following.

THEOREM B. *The Banach fixed point principle implies Theorem A.*

P r o o f. Let (S',F,T) and f be as in Theorem A. If p_o is given in S' then let a_n and b_n be as in Lemma 2 and such that $F_{p_o f p_o}(a_n) \geq b_n$. Consider the generalized metric d as in Lemma 2. It is each to see that $d(p_o, fp_o) < \infty$ and therefore [5] $S_o = \{q_o \in S,\ d(p_o,q_o) < \infty\}$ is a complete <u>metric</u> space and f is a contraction in S_o. Therefore $p_n = f^n p_o$ d-converges to the (evidently unique) fixed point of f and the theorem follows.

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REZIME

O t-NORMAMA HADŽIĆ TIPA I NEPOKRETNE TAČKE U VEROVATNOSNIM METRIČKIM PROSTORIMA

Dat je nov dokaz rezultata iz [4] i [6] i ispitana struktura h-t-normi.

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A NOTE ON ALMOST CLOSED MAPPINGS AND NEARLY PARACOMPACTNESS

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ABSTRACT

In this note it will be shown that the Lemma 1.1 in [4] and the proofs of some theorems in [3], [4], [7] and [10], where this lemma was used, are not correct.

It will be shown that this lemma and these theorems are correct with new additional condition.

Moreover, some new characterizations of almost closed mappings will be shown.

All definitions could be find in the paper [4].

In [11] T.Noiri has proved:

LEMMA A. *If a mapping* $f:X\to Y$ *is almost continuous and almost open, then:*

a) For each regularly open set V *of* Y, $f^{-1}(V)$ *is regularly open in* X,

b) For each regularly closed set V *of* Y, $f^{-1}(V)$ *is regularly closed in* X.

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In [3] the author has proved:

LEMMA B. *(Lemma 1.1) If a mapping* $f:X\to Y$ *is almost continuous and almost closed, then:*

a) For each regularly closed set F *of* $Y, f^{-1}(F)$ *is regularly closed in* X,

b) For each regularly open set V *of* $Y, f^{-1}(V)$ *is regularly open open in* X.

From the following example it follows that Lemma B is not correct.

EXAMPLE 1. Let

$$X=\{a,b,c,d,e\},\quad \tau_X=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\},X\};$$

$$Y=\{a,b,c\},\qquad \tau_Y=\{\emptyset,\{a\},\{b\},\{a,b\},Y\}.$$

Let $f:X\to Y$ be a mapping of a space X onto a space Y defined by

$$f(a)=b, f(b)=a,\quad f(c)=f(d)=f(e)=c.$$

f is almost continuous and almost closed. $\{b\}$ is regularly open in Y, since $\alpha(\{b\})=\{\bar{b}\}^{o}=\{b,c\}^{o}=\{b\}$. But $f^{-1}(\{b\}) =$ $=\{a\}$ is not regularly open in X, since

$$\alpha(\{a\})=\{a,c,d,e\}^{o}=\{a,c\}\neq\{a\}.$$

However, we can show that the Lemma B is necessarily true if a new condition is added.

LEMMA 1. *If* $f:X\to Y$ *is almost continuous and almost closed surjection, such that for each regularly closed set* F *of* Y

$$f^{-1}(f(\overline{[f^{-1}(F)]^{o}}))=\overline{[f^{-1}(F)]^{o}}\quad,\qquad \textit{then}$$

a) For each regularly closed set F *of* $Y, f^{-1}(F)$ *is regularly closed in* X;

b) For each regularly open set V *of* $Y, f^{-1}(V)$ *is regularly open in* X.

P r o o f. a) Let F be any regularly closed subset of Y. Then $f^{-1}(F)$ is closed. Hence we have

$$\overline{[f^{-1}(F)]^{o}} \subset f^{-1}(F) .$$

On the other hand, since f is almost countinuous and F^{o} is non empty regularly open subset of Y, $f^{-1}(F^{o})$ is open, hence

$$f^{-1}(F^{o}) \subset [f^{-1}(F)]^{o} \subset \overline{[f^{-1}(F)]^{o}} .$$

Since f is almost closed and $\overline{[f^{-1}(F)]^{o}}$ is regularly closed, then $f(\overline{[f^{-1}(F)]^{o}})$ is closed. Since $f^{-1}(F^{o}) \subset \overline{[f^{-1}(F)]^{o}}$, then

$$F^{o} \subset f(\overline{[f^{-1}(F)]^{o}}) \text{ , i.e. } F = \overline{F^{o}} \subset f(\overline{[f^{-1}(F)]^{o}}) .$$ Hence we

have $$f^{-1}(F) \subset f^{-1}(f(\overline{[f^{-1}(F)]^{o}})) = \overline{[f^{-1}(F)]^{o}} .$$

Hence we have

$$f^{-1}(F) = \overline{[f^{-1}(F)]^{o}} .$$

b) Let U be any regularly open subset of Y. Then $Y \setminus U$ is a regularly closed subset of Y. $f^{-1}(Y\setminus U) = X\setminus f^{-1}(U)$ is regularly closed, hence $f^{-1}(U)$ is regularly open.

A mapping $f:X \to Y$ is said to be α-irreducible iff for every regularly closed subset F of Y there is no proper regularly subset of $f^{-1}(F)$ mapped onto the whole of F.

COROLLARY 1. *If f is an almost closed, almost continuous and α-irreducible mapping of a space X onto a space Y, then for each regularly open (regularly closed) subset F of Y, $f^{-1}(F)$ is regularly open (regularly closed) in X .*

The following example shows that there exists a mapping with the properties as in Lemma 1, which is not almost open.

EXAMPLE 2. Let

$$X = \{a,b,c,d\} \text{ , } \tau_X = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},$$
$$\{a,b,c\},X\} \text{ ;}$$
$$Y = \{a,b,c\}, \quad \tau_Y = \{\emptyset,\{a\},\{c\}.\{a,c\},Y\} .$$

Let $f:X \to Y$ be a mapping of a space X onto a space Y defined by:

$$f(a) = a, \; f(b) = f(d) = b, \; f(c) = c \ .$$

f is almost closed and almost continuous. f is not almost open, since $f(\{b\}) = \{b\}$ ($\{b\}$ is a regularly open set in X, since $\alpha(\{b\}) = \{b,d\}^{o} = \{b\}$) is not open in Y.

The proper regularly closed subsets in the space Y are the subsets

$$F_1 = \{a,b\} \quad \text{and} \quad F_2 = \{b,c\} \ .$$

$$\overline{[f^{-1}(F_1)]^{o}} = \overline{\{a,b,d\}^{o}} = \{a,b,d\}; \quad \overline{[f^{-1}(F_2)]^{o}} = \overline{\{b,c,d\}^{o}} = \{b,c,d\}.$$

$$f^{-1}(f(\overline{[f^{-1}(F_1)]^{o}})) = f^{-1}(f(\{a,b,d\})) =$$

$$= f^{-1}(\{a,b\}) = \{a,b,d\} \ ;$$

$$f^{-1}(f(\overline{[f^{-1}(F_2)]^{o}})) = f^{-1}(f(\{b,c,d\})) = f^{-1}(\{b,c\}) = \{b,c,d\} \ .$$

f is not α-irreducible, since $\overline{f^{-1}(F_1^{o})} = \{a,d\}$ is a proper regularly closed subset of $f^{-1}(F_1) = \{a,b,d\}$ such that $f(\overline{f^{-1}(F_1^{o})}) = F_1$.

The following example shows that there exists a mapping with the properties as in Lemma 1, which is not almost open and continuous.

EXAMPLE 3. Let

$X = \{a,b,c,d,e\}$, $\tau_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$;

$Y = \{a,b\}$, $\tau_Y = \{\emptyset, \{a\}, Y\}$.

Let $f:X \to Y$ be a mapping of a space X onto a space Y defined by:

$$f(a) = f(d) = a, \qquad f(c) = f(b) = f(e) = b \ .$$

f is almost closed and almost continuous. f is not almost open, since $f(\{b\}) = \{b\}$, nor continuous, since $f^{-1}(\{a\}) = \{a,d\}$.

By using Lemma B, the author has proved:

Theorem A ([3], [4], [7]). Let $f:X\to Y$ be any almost closed, almost continuous mapping of a space X onto a space Y. Then:

a) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly paracompact (α-nearly compact) and if X is almost regular, Y is almost regular.

b) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly compact and if X is almost regular nearly paracompact, Y is almost regular nearly paracompact,

c) if K is α-nearly compact, f(K) is α-nearly compact,

d) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly compact and if X is a Hausdorff locally nearly compact space, Y is Hausdorff locally nearly compact.

The proof of this Theorem is not correct, since we are used that the inverse image of every regularly open set is regularly open. We do not know if the formulation of this Theorem is true.

However, we can show that Theorem B is necessarily true if a new condition is added.

THEOREM 1. *Let $f:X\to Y$ be any almost closed, almost continuous mapping of a space X onto a space Y, such that for every regularly closed set F of Y,* $f^{-1}(\overline{f([f^{-1}(F)]^{o})})=[f^{-1}(F)]^{o}$ *. Then:*

a) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly paracompact and if X is almost regular, Y is almost regular,

b) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly compact and if X is almost regular nearly paracompact, Y is almost regular nearly paracompact,

c) if K is α-nearly compact, f(K) is α-nearly compact,

d) if for each point $y\in Y$, $f^{-1}(y)$ is α-nearly compact and if X is a Hausdorff locally nearly compact space, Y is Hausdorff locally nearly compact.

P r o o f. a) It is identical with the proof of Theorem 2.3 in [4].

b) It is identical with the proof of Theorem 2.1 in [3].

c) It is identical with the proof of Lemma 2.1 in [4].

d) It is identical with the proof of Theorem 2.9 in [4].

REMARK 1. Theorem 1 is true if $f: X \to Y$ is an almost closed, almost continuous and α-irreducible surjection.

THEOREM 2. *If* f *is a closed almost continuous mapping of a regular space* X *onto a space* Y *such that* $f^{-1}(y)$ *is* α*-paracompact for each point* $y \in Y$*, then* Y *is almost regular.*

P r o o f. It is similar to the proof of Theorem 2.2 in [4].

COROLLARY 2. *If* f *is an almost closed almost continuous mapping of a regular space* X *onto a space* Y*, such that* $f^{-1}(y)$ *is* α*-nearly paracompact for each point* $y \in Y$*, then* Y *is almost regular.*

P r o o f. f is closed (Theorem 1, [9]). In a regular space every α-nearly paracompact is α-paracompact.

COROLLARY 3. *([11]) If* X *is regular and* $f: X \to Y$ *is an almost continuous and almost closed surjection such that* $f^{-1}(y)$ *is compact for each point* $y \in Y$*, then* Y *is almost regular.*

THEOREM 3. *If* f *is an almost closed continuous mapping of an almost normal space* X *onto a space* Y *such that for each regularly closed subset* F *of* Y*,* $f^{-1}(f(\overline{[f^{-1}(F)]^{o}})) = \overline{[f^{-1}(F)]^{o}}$*, then* Y *is almost normal.*

P r o o f. Let U be an open and V an regularly open set in Y such that $U \cup V = Y$. Then $f^{-1}(U)$ is open and $f^{-1}(V)$ is a regularly open set in X such that $f^{-1}(U) \cup f^{-1}(V) = X$. Since X is almost normal, then by Lemma 2.1 in [5], there exist the regularly closed subsets A_o and B_o of X such that $A_o \subset f^{-1}(U)$, $B_o \subset f^{-1}(V)$ and $A_o \cup B_o = X$. Since f is almost closed, then $A = f(A_o)$ and $B = f(B_o)$ are closed sets, such that $A \subset U$, $B \subset V$

and $A \cup B = Y$. Hence, by Lemma 2.1 in [5], Y is almost normal.

LEMMA 2. *Let f be any mapping of an almost regular space X onto a space Y such that for each point $y \in Y$, $f^{-1}(y)$ is α-nearly paracompact, then the following are equivalent:*

a) f is almost closed,

b) f is star closed,

c) for any subset K in Y and any star open set U containing $f^{-1}(K)$, there exists an open set V in Y such that $K \subset V$ and $f^{-1}(V) \subset U$.

P r o o f. (a) → (b). First, it will be shown that $f:(X,\tau^*) \to Y$ is almost closed (τ^* is semiregularization of τ). Let F be any τ^*-regularly closed set. Then, there exists τ^*-open set U such that $F = \bar{U}_{\tau^*} = \bar{U}_{\tau}$, hence F is τ-regularly closed. Since $f:(X,\tau) \to Y$ is almost closed, then $f(F)$ is closed. f is the almost closed mapping of the regular space (X,τ^*) onto a space Y such that $f^{-1}(y)$ is α-paracompact for each point $y \in Y$, hence, by Theorem 1 in [9] $f:(X,\tau^*) \to Y$ is closed. Hence the implication is proved.(b) → (a), which is obvious.

(b) → (c). Let K be any subset in Y and U be any star open subset in X containing $f^{-1}(K)$. Let $V = Y \setminus f(X \setminus U)$. Since f is star closed, $f(X\setminus U)$ is closed. Hence V is open in Y such that

$$f^{-1}(K) \subset f^{-1}(V) = X \setminus f^{-1}(f(X\setminus U)) \subset U .$$

(c) → (b). Let A be any star closed subset in X and $y \in Y \setminus f(A)$ be any point. Then we have $f^{-1}(y) \subset X \setminus A$. Since $X\setminus A$ is star closed, there exists an open set V in Y such that $f^{-1}(y) \subset f^{-1}(V) \subset X\setminus A$, i.e. $y \in Y \subset Y \setminus f(A)$, hence $Y\setminus f(A)$ is open in Y. Hence $f(A)$ is closed.

LEMMA 3. *Let X be almost regular, and A and B be any disjoint sets such that A is α-nearly paracompact and B star closed. Then, there exist disjoint regularly open sets containing A and B respectively.*

P r o o f. $X\setminus B$ is τ^*-open set containing A which is α-paracompact in (X,τ^*). Since (X,τ^*) is regular, there exists a τ^*-open set C such that $A\subset C\subset \bar{C}_{\tau^*}=\bar{C}_{\tau}\subset X\setminus B$. Let $U=\alpha(C)$. Then, U is a regularly open set such that $A\subset U\subset \bar{U}\subset X\setminus B$. Let $V=X\setminus U$. U and V are disjoint regularly open sets containing A and B respectively.

COROLLARY 4. *Let X be any Hausdorff almost regular space. Then, for any disjoint α-nearly paracompact sets A and B, there exist the disjoint regularly open sets U and V containing A and B respectively.*

P r o o f. In a Hausdorff space every α-nearly paracompact is star closed (Theorem 1. [7]).

THEOREM 4. *If $f:X\to Y$ is an almost closed mapping of an almost regular space X onto a space Y such that $f^{-1}(y)$ is star closed for each point $y\in Y$ and $f^{-1}(F)$ is α-nearly paracompact for each proper regularly closed set F of Y, then Y is almost regular.*

P r o o f. Let F be any regularly closed set in Y and $y\notin F$ be any point. Then, by the preceding Lemma, there exist the disjoint regularly open sets U and V such that $f^{-1}(F)\subset U$ and $f^{-1}(y)\subset V$. Since f is almost closed, then there exist the open sets U_1 and V_1 in Y such that $F\subset U_1$, $y\in V_1$, $f^{-1}(F)\subset f^{-1}(U_1)\subset U$ and $f^{-1}(y)\in f^{-1}(V_1)\subset V$. Hence, Y is almost regular.

COROLLARY 5. *If $f:X\to Y$ is an almost closed mapping of a Hausdorff almost regular space X onto a space Y such that $f^{-1}(y)$ and $f^{-1}(F)$ are α-nearly paracompact for each point $y\in Y$ and each proper regularly closed set F in Y, then Y is Hausdorff almost regular.*

P r o o f. Y is almost regular. Now, we shall show that Y is Hausdorff. Let a and b be different points of Y. $f^{-1}(a)$ and $f^{-1}(b)$ are disjoint α-nearly paracompact sets in X, hence there exist the disjoint regularly open sets U and V containing $f^{-1}(a)$ and $f^{-1}(b)$ respectively.

Since f is almost closed, then there exist open sets U_1 and V_1 such that $a \in U_1$, $b \in V_1$, $f^{-1}(U_1) \subset U$ and $f^{-1}(V_1) \subset V$, hence Y is Hausdorff.

THEOREM 5. *If f is an almost closed mapping of a Hausdorff space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$ and $f^{-1}(F)$ is α-nearly paracompact for each proper regularly closed set F of Y, then Y is Hausdorff almost regular.*

P r o o f. By the Theorem 3 in [13] Y is Hausdorff. Now, we shall show that Y is almost regular. Let F be any regularly closed subset of Y and $y \notin F$ any point. Since $f^{-1}(y)$ is α-nearly compact and $f^{-1}(F)$ is α-nearly paracompact, then by Theorem 2.1 in [4] there exist disjoint regularly open sets U an V containing $f^{-1}(y)$ and $f^{-1}(F)$ respectively. Since f is almost closed, there exist disjoint open sets U_1 and V_1 containing F and y respectively, hence Y is almost regular.

THEOREM 6. *If $f:X \to Y$ is an almost closed and almost continuous mapping of a Hausdorff nearly paracompact (almost regular nearly paracompact) space X onto a space Y such that for each proper regularly closed set F of Y, $f^{-1}(F)$ is α-nearly paracompact and for each point $y \in Y$, $f^{-1}(y)$ is α-nearly compact (star closed α-nearly compact) then, Y is Hausdorff almost regular nearly paracompact.*

P r o o f. Y is Hausdorff almost regular. We shall show that Y is nearly paracompact. Let $\mathcal{U} = \{U_i : i \in I\}$ be any regularly open covering of Y. Since f is almost continuous, $f^{-1}(\mathcal{U}) = \{f^{-1}(U_i) : i \in I\}$ is open covering of a space X. Since X is nearly paracompact, there exists regularly open locally finite refinement $\mathcal{V} = \{V_j : j \in J\}$ of $\{\alpha(f^{-1}(U_i)) : i \in I\}$.Then by Lemma 2 in [12] $\{f(V_j) : j \in J\}$ is locally finite covering of Y. $\{\overline{f(V_j)} : j \in J\}$ is closed locally finite covering of Y. For each $j \in J$ there exists $i(j) \in I$ such that $V_j \subset \alpha(f^{-1}(U_{i(j)})) \subset f^{-1}(\overline{U_{i(j)}})$, hence $\overline{f(V_j)} \subset f(\overline{V_j}) \subset \overline{U_{i(j)}}$.

Now, $\{\overline{f(V_j)}: j \in J\}$ is a closed locally finite refinement of $\{\bar{U}_i: i \in I\}$.

By Lemma 1.1 in [20], the family $\{\overline{[\overline{f(V_j)}]^o}: j \in J\}$ is a locally finite regularly closed cover of X. Since $[\overline{f(V_j)}]^o \subset \alpha(U_{i(j)}) = U_{i(j)}$, then $\{[\overline{f(V_j)}]^o: j \in J\}$ is a locally finite family of open sets which refines $\mathcal{U}$ and the closures of whose members cover the space Y. Hence, by Lemma 1.3 in [6], Y is almost paracompact. Since every almost regular almost paracompact is nearly paracompact, then Y is nearly paracompact.

THEOREM 7. *If* $f:X \to Y$ *is an almost closed mapping of a Hausdorff locally nearly compact space onto a space* Y *such that* $f^{-1}(F)$ *is* α*-nearly paracompact for each proper regularly closed subset* $F \subset Y$ *and* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$, *then* Y *is a Hausdorff locally nearly compact.*

P r o o f. Y is a Hausdorff almost regular space. We shall show that Y is locally nearly compact. Since X is a Hausdorff nearly compact, for each point $x \in f^{-1}(y)$, there exists a regularly open neighbourhood K_x such that $\bar{K}_x$ is α-nearly compact. Now, the family

$$\mathcal{K} = \{K_x: x \in f^{-1}(y)\}$$

is an X–regularly open cover of $f^{-1}(y)$, hence there exist a finite number of points $x_1, x_2, \ldots, x_n$ in $f^{-1}(y)$ such that

$$f^{-1}(y) \subset \bigcup\{K_{x_i}: i=1,2,\ldots,n\} .$$

Let

$$K = \bigcup\{\bar{K}_{x_i}: i=1,2,\ldots,n\} .$$

$f^{-1}(y) \subset K^o$. Since f is almost closed, then there exists an open set V_y containing y such that $f^{-1}(V_y) \subset K^o$. Hence, we have

$$y \in V_y \subset f(K^o) \subset f(K) .$$

Since in a Hausdorff space every α-nearly paracompact is star closed, then f is almost continuous. Since f is almost continuous and almost closed and K is α-nearly compact, then f(K)

is almost compact, i.e. α-nearly compact (X is almost regular). f(K) is closed, hence $\bar{V}_y \subset f(K)$. $\bar{V}_y$ is α-nearly compact, hence Y is locally nearly compact.

By using Lemma B the author has proved:

THEOREM B. ([4], [10]) *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *and let* $\mathcal{D}$ *have a quotient topology. Then:*

a) if X *is almost normal,* $\mathcal{D}$ *is almost normal ;*

b) if the members of $\mathcal{D}$ *are α-nearly paracompact subsets of* X *and if* X *is almost regular,* $\mathcal{D}$ *is almost regular;*

c) if the members of $\mathcal{D}$ *are α-nearly compact subsets of* X *and if* X *is almost regular (almost regular nearly paracompact Hausdorff locally nearly compact)* $\mathcal{D}$ *is almost regular (almost regular nearly paracompact, Hausdorff locally nearly compact).*

The proof of this Theorem is not correct, since we have used that the inverse image of every regularly open set is regularly open.

EXAMPLE 4. Let

$$X = \{a,b,c,d,e\},\ \tau_X = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}, X\} ;$$

$$\mathcal{D} = \{\{a\},\{b\},\{c,d,e\}\};\ \tau_{\mathcal{D}} = \{\emptyset, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \mathcal{D}\},\ R = \bigcup\{D \times D : D \in \mathcal{D}\} .$$

The projection $P: X \to X/R$ is almost closed (the decomposition is almost-upper semicontinuous) such that $P^{-1}(\{\{a\}\}) = \{a\}$ is not regularly open in X.

LEMMA 4. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *such that for each proper regularly closed set* A *of* X, R[A] *is α-nearly paracompact* ($R = \bigcup\{D \times D : D \in \mathcal{D}\}$) . *Let* $\mathcal{D}$ *have a quotient topology. Then, for each proper regularly closed subset* A *of* $\mathcal{D}$, $P^{-1}(A)$ *is α-nearly paracompact.*

P r o o f. Let A be any proper regularly closed subset of $\mathcal{D}$. Then we have $P^{-1}(A^{o}) \subset \overline{[P^{-1}(A)]^{o}} \subset P^{-1}(A)$, hence $P(P^{-1}(A^{o})) \subset$ $\subset P(\overline{[P^{-1}(A)]^{o}}) \subset P(P^{-1}(A))$, i.e. $A^{o} \subset P(\overline{[P^{-1}(A)]^{o}}) \subset A$. Since $P(\overline{[P^{-1}(A)]^{o}})$ is closed we have $A = P(\overline{[P^{-1}(A)]^{o}})$, i.e.

$$P^{-1}(A) = P^{-1}(P(\overline{[P^{-1}(A)]^{o}})) = R[\overline{P^{-1}(A)^{o}}]$$

Since $\overline{[P^{-1}(A)]^{o}}$ is regularly closed, hence $P^{-1}(A)$ is α-nearly paracompact.

THEOREM 8. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X. *Let* $\mathcal{D}$ *have a quotient topology. Then:*

a) if the members of $\mathcal{D}$ *are star closed in* X *such that for each proper regularly closed set* A *of* X, $R[A]$ *is* α*-nearly paracompact* ($R = \bigcup\{D \times D \colon D \in \mathcal{D}\}$) *and if* X *almost regular,* $\mathcal{D}$ *is almost regular ;*

b) if the members of $\mathcal{D}$ *are* α*-nearly paracompact in* X *such that for each proper regularly closed set* A *of* X, $R[A]$ *is* α*-nearly paracompact and if* X *is Hausdorff almost regular, then* $\mathcal{D}$ *is Hausdorff almost regular;*

c) if the members of $\mathcal{D}$ *are* α*-nearly compact of* X *such that for each proper regularly closed set* A *of* X, $R[A]$ *is* α*-nearly paracompact and if* X *is Hausdorff locally nearly compact (Hausdorff nearly paracompact),* $\mathcal{D}$ *is Hausdorff almost regular locally nearly compact (Hausdorff almost regular nearly paracompact);*

d) if the members of $\mathcal{D}$ *are star closed* α*-nearly compact of* X *such that for each proper regularly closed set* A *of* X, $R[A]$ *is* α*-nearly paracompact and if* X *is almost regular nearly paracompact,* $\mathcal{D}$ *is Hausdorff almost regular nearly paracompact.*

P r o o f. a) This follows from Theorem 4.

b) This follows from Corollary 5.

c) This follows from Theorem 6 and Theorem 7.

d) This follows from Theorem 6.

THEOREM 9. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X, such that for each proper regularly closed subset A of X,* $(R = \bigcup\{D \times D : D \in \mathcal{D}\})$ $R[A] = \{R[x] : x \in A\}$ *is locally finite. Let $\mathcal{D}$ have a quotient topology. Then:*

a) if the members of $\mathcal{D}$ are closed in X then a subset A of $\mathcal{D}$ is regularly open iff it is regularly closed;

b) if the members of $\mathcal{D}$ are closed in X then $P^{-1}(A)$ is regularly open (regularly closed) for every regularly open (regularly closed) subset of $\mathcal{D}$,

c) if the members of $\mathcal{D}$ are closed in X and if X is almost normal, $\mathcal{D}$ is almost normal;

d) if X is Hausdorff locally nearly compact (Hausdorff nearly paracompact) and if the members of $\mathcal{D}$ are α-nearly compact in X, $\mathcal{D}$ is Hausdorff almost regular locally nearly compact (Hausdorff almost regular nearly paracompact);

e) if the members of $\mathcal{D}$ are closed α-nearly paracompact in X and if X is almost regular, $\mathcal{D}$ is almost regular.

P r o o f. a) Let F be any regularly closed subset of $\mathcal{D}$. Then $\overline{[P^{-1}(F)]^{o}}$ is regularly closed, such that $P^{-1}(P(\overline{[P^{-1}(F)]^{o}})) = P^{-1}(F) = R[\overline{P^{-1}(F)^{o}}] = \bigcup\{R[x] : x \in \overline{[P^{-1}(F)]^{o}}\}$. Since $\{R[x] : x \in \overline{[P^{-1}(F)]^{o}}\}$ is locally finite then, the family $\{R[x] : x \in P^{-1}(F^{o})\}$ is locally finite. Thus, we have $\overline{P^{-1}(F^{o})} = \bigcup\{\overline{R[x]} : x \in P^{-1}(F^{o})\} = P^{-1}(F^{o})$. Hence, F is open and closed.

b) This is obvious.

c) This is similar to the proof Theorem 3.

d) This is similar to the proof of b) and d) in Theorem 1.

e) This is similar to the proof of a) in Theorem 1.

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REZIME

SKORO ZATVORENA PRESLIKAVANJA U BLIZU PARAKOMPAKTNOST

U radu se posmatraju osobine skoro zatvorenih preslikavanja. Pokazuje se da postoji skoro zatvoreno i skoro neprekidno preslikavanje sa osobinom da inverzna slika skoro otvorenog (skoro zatvorenog) skupa nije uvek skoro otvoren (skoro zatvoren) skup. Medjutim, ako je f skoro zatvoreno i skoro neprekidno preslikavanje prostora X na prostor Y sa osobinom da je $f^{-1}(f([f^{-1}(F)]^{o}))= [f^{-1}(F)]^{o}$ tada je inverzna slika svakog skoro otvorenog (skoro zatvorenog) skupa skoro otvoren (skoro zatvoren) skup.

Dalje se ispituje kako se odnose skoro regularni, Hausdorffovi, skoro normalni i blizu parakompaktni prostori pri skoro zatvorenim preslikavanjima.

Na kraju se daju neke osobine skoro odozgo poluneprekidnog razlaganja datog prostora.

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ON A CLASS OF FUNCTIONS

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ABSTRACT

In [3] H. Steinhaus introduced the concept of a permutation function of the interval $[0,1)$ and proved several theorems about these functions. A. Mookhopadhyaya [2] and H. Miller [1] each have several results dealing with Steinhaus permutation functions.

The purpose of this paper is to consider another class of functions, which we will call "switch functions", and show that they share certain properties with the Steinhaus permutation functions.

1. PRELIMINARIES. Each number $t \in [0,1)$, can be writen in one and only one way in dyadic form. This is not quite true that is some numbers, those having finite dyadic representations, have two representations; for example

$$1/2 = \sum_{n=1}^{\infty} 1/2^{n+1}.$$

In such cases we always assign the finite representation to the number under consideration, and in this sense each number in $[0,1)$ has one and only one representation.

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By a permutation P of the natural numbers we shall mean any one-to-one function of the natural numbers onto themselves.

As mentioned above each $t \in [0,1)$ can be written in the form

$$t = \sum_{n=1}^{\infty} (e_n/2^n) ,$$

where each e_n is either 0 or 1.

For simplicity we will write this formula as

$$t = 0 \cdot e_1 e_2 e_3 \ldots .$$

If P is a permutation, then by applying P to the indices on the right hand side of the equation for t we obtain a new development which corresponds to a real number t' in $[0,1)$ given by

$$t' = 0 \cdot e_1' e_2' e_3' \ldots \quad , \text{ where } e_n' = e_{P(n)}$$

for each $n = 1, 2, 3, \ldots$.

For convenience we shall write $P(t) = t'$. That is the same symbol, i.e. P, is used for two different functions, but no confusion will occur as it will always be clear which P we are discussing from the context in which it occurs.

Many facts about Steinhaus permutations functions (i.e. functions of the type $P : [0,1) \to [0,1)$, $P(t) = t'$ described above) are known. In the following some of the most important facts about Steinhaus permutation functions will be listed.

1) If E is a Lebesgue measurable subset of $[0,1)$ and P is any Steinhaus permutation function, then

$$P(E) = \{ P(e) : e \in E \}$$

is a Lebesgue measurable set and

$$m(P(E)) = m(E) ,$$

where m denotes Lebesgue measure. Two different proofs of this fact can be found in the literature (H.Steinhaus [3]

and H.Miller [1]).

2) Each Steinhaus permutation function is continuous at each point of $[0,1)$ with the exception of at most countabily many points. The proof of this theorem is given in the paper of A.Mookhopadyaya [2].

3) Suppose that P is a Steinhaus permutation function that moves infinitely many natural numbers (i.e. $P(n) \neq n$ for infinitely many n). Then it follows that $P'(x)$, the derivative of P at x, exists nowhere on $[0,1)$. This result is due to H.Miller and can be found in [1].

We will now consider a new class of functions which we will call switch functions.

If $t, y \in [0,1)$ and

$$t = 0\cdot e_1 e_2 e_3 \ldots ,$$

$$y = 0\cdot y_1 y_2 y_3 \ldots$$

are the unique (adopting the earlier mentioned convention) binary developments of t and respectively, then $S_y(t)$ is defined by the formula

$$S_y(t) = 0\cdot e_1' e_2' e_3' \ldots ,$$

where

$$e_n' = e_n \quad \text{if} \quad y_n = 0 \quad \text{and}$$

$$e_n' = \hat{e}_n \quad \text{if} \quad y_n = 1,$$

where ^ is the switch operation, i.e. $\hat{0} = 1$ and $\hat{1} = 0$. The function $t \to S_y(t)$ (having domain $[0,1)$) will be called the switch function determined by y.

In this paper switch function analogues of results about Steinhaus permutation functions will be proved.

2. RESULTS. Let $\mathcal{P}$ denote the collection of all Steinhaus permutation functions and let $\mathcal{S}$ denote the collection of all switch functions. In our first result we will show that the only function in $\mathcal{P} \cap \mathcal{S}$ is the identity function defined on $[0,1)$.

THEOREM 1. $P \cap \mathcal{S} = \{i\}$, *where* i *denotes the identity function on* $[0,1)$, *i.e.* $i(x) = x$ *for every* $x \in [0,1)$

P r o o f. Suppose that P is a permutation of the natural numbers and $P(i) = j$, with $i \neq j$.

Then $(P(x))_i$, the i^{th} number in the expansion of $P(x)$, is given by

$$(P(x))_i = x_j \ , \quad \text{where } x = 0\cdot x_1 x_2 x_2 \ \ldots \ .$$

Furthermore $(S_y(x))_i$, the i^{th} number in the expansion of $S_y(y)$, is given by

$$(S_y(x))_i = x_i \quad \text{or} \quad \hat{x}_i \ ,$$

depending on whether y_i is 0 or 1, where $y = 0\cdot y_1 y_2 y_3, \ldots$.

In any case by the statistical independence of the numbers appearing in the i^{th} and j^{th} places of the binary developments of numbers in $[0,1)$ we have

$$m(x \in [0,1) : (P(x))_i \neq (S_y(x))_i) = 1/2 \ .$$

From this it follows that $P \neq S_y$ for every $y \in [0,1)$.

We next prove the analogue of 2) in section 1.

THEOREM 2. *If* $S \in \mathcal{S}$, *then* S *is continuous on* $[0,1)$ *with the exception of at most countabily many points.*

P r o o f. If $S \in \mathcal{S}$, then $S = S_y$ for some $y \in [0,1)$. Consider the sequence of functions $(S_y^n)_{n=1}^{n=\infty}$ defined as follows. For each $x = 0\cdot x_1 x_2 x_3 \ \ldots$ and each natural number n

$$((S_y^n)(x))_i = (S_y(x))_i \quad \text{for} \quad i = 1, 2, \ldots, n$$

and

$$((S_y^n)(x))_i = x_i \quad \text{for all} \quad i > n.$$

Then the sequence $(S_y^n)_{n=1}^{n=\infty}$ has the following properties.

a) $(S_y^n)'(x)$, the derivative of (S_y^n) at x, equals one for each $x \in [0,1)\setminus C_n$, where C_n is a finite set for each n.

b) The sequence $(S_y^n)_{n=1}^{n=\infty}$ converges uniformly to S_y on $[0,1)$.

From a) and b) it is immediate that S_y is continuous at each point of the set $[0,1)\setminus \bigcup_{n=1}^{\infty} C_n$.

The next result is an analogue of 1) in section 1.

THEOREM 3. *If* E *is a Lebesgue measurable subset of* $[0,1)$ *and* S_y *is any switch function, then*

$$S_y(E) = \{S_y(e) : e \in E\}$$

is a Lebesgue measurable set and

$$m(S_y(E)) = m(E) .$$

P r o o f. Let the sequence $(S_y^n)_{n=1}^{n=\infty}$ be defined as in the proof of Theorem 2. Since $(S_y^n)'(x)$ exists for all x in $[0,1)\setminus C_n$, where C_n is a finite set, it follows that each function S_y^n is Lebesgue measurable (in fact is a Baire function of class one). Therefore, by b) in the proof of Theorem 2, it follows that S_y is Lebesgue measurable (in fact is a Baire function of class two). Let B be any Borel subset of $[0,1)$. Define

$$(S_y)^{-1}(B) = (x \in [0,1) : S_y(x) \in B) .$$

It is not difficult to see that the symmetric difference of the sets

$$S_y(B) \quad \text{and} \quad (S_y)^{-1}(B) \cdot$$

is an at most countable set, where the symmetric difference of any two sets M and N is defined to be the set $(M\setminus N) \cup (N\setminus M)$. Therefore $S_y(B)$ is a Lebesgue measurable set for each Borel

set B (in fact $S_y(B)$ is a Borel set).

The remainder of the proof follows the proof of Theorem 2 in [1], but is included for completeness.

By Theorem 2, S_y is continuous on a set $[0,1)\setminus C$, where C is at most countable. Let B' denote the set $B\setminus C$. Clearly $m(B')=m(B)$. Furthermore, for every $\varepsilon>0$, there exists C_ε, a closed subset of B', such that

$$m(B') - m(C_\varepsilon) < \varepsilon .$$

We will now show that for each $x\in[0,1)$,

$$\limsup_{n\to\infty} X_{S_y^n(C_\varepsilon)}(x) \le X_{S_y(C_\varepsilon)}(x) .$$

Here X_D denotes the characteristic function of the set D, i.e. $X_D(x)=1$ if $x\in D$ and $X_D(x)=0$ if $x\notin D$. To see this suppose that $x\in[0,1)$ and $\limsup_{n\to\infty} X_{S_y^n(C_\varepsilon)}(x)=1$. Then there exists a subsequence $(n_k)_{k=1}^{k=\infty}$ of the positive integers, with $x\in S_y^{n_k}(C_\varepsilon)$ for each integer k. Therefore for each k there exists $e_k\in C_\varepsilon$, such that $x=S_y^{n_k}(e_k)$. There is a subsequence $(e_{k_j})_{j=1}^{j=\infty}$ of the sequence $(e_k)_{k=1}^{k=\infty}$ such that the $\lim_{j\to\infty} e_{k_j}$ exists, that is $\lim_{j\to\infty} e_{k_j}=e$.

However C_ε is a closed set and therefore $e\in C_\varepsilon$. This in turn implies that

$$S_y(e) = \lim_{j\to\infty} S_y^{n_{k_j}}(e_{k_j}) ,$$

since the sequence $(S_y^n)_{n=1}^{n=\infty}$ converges uniformly to S_y on $[0,1)$, e is a point of continuity of S_y and $\lim_{j\to\infty} e_{k_j}=e$. But $x=S_y^{n_{k_j}}(e_{k_j})$ for each $j=1,2,\dots$.

Therefore $S_y(e)=x$, with $e\in C_\varepsilon$ and hence

$$X_{S_y(C\ )} = 1.$$

Therefore we have shown that for each $x \in [0,1)$,

$$\limsup_{n\to\infty} X_{S_y^n(C_\varepsilon)}(x) \le X_{S_y(C_\varepsilon)}(x) .$$

This implies that $\lim_{n\to\infty} f_n(x) \le X_{S_y(C_\varepsilon)}(x)$ for every $x \in [0,1)$, where

$$f_n(x) = \sup_{k\ge n} X_{S_y^k(C_\varepsilon)}(x) .$$

Therefore $\lim_{n\to\infty} (f_n(x) - X_{S_y(C_\varepsilon)}(x)) \le 0$ and by the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \le \int_0^1 X_{S_y(C_\varepsilon)}(x)\,dx .$$

Furthermore

$$\int_0^1 f_n(x)\,dx \ge \int_0^1 X_{S_y^n(C_\varepsilon)}(x)\,dx = m(S_y^n(C_\varepsilon)).$$

By a) in the proof of Theorem 2 it follows that

$m(S_y^n(C_\varepsilon)) = m(C_\varepsilon)$ and therefore we have

$$m(S_y(C_\varepsilon)) = \int_0^1 X_{S_y(C_\varepsilon)}(x)\,dx \ge m(C_\varepsilon) , \quad \forall\varepsilon > 0.$$

From this it follows that

$$m(S_y(B)) \ge m(B)$$

for every Borel set B contained in $[0,1)$.

If $m(S_y(B)) > m(B)$ for some Borel subset of $[0,1)$, then we would have

$$m(S_y(B)) + m(S_y(B^c)) > m(B) + m(B^c) ,$$

where $B^c = [0,1) \setminus B$; which implies $1 > 1$.

Therefore we have shown that

$$m(S_y(B)) = m(B)$$

for every Borel subset B of $[0,1)$.

Finally if E is any measurable subset of $[0,1)$ then there exist Borel sets B_1 and B_2, such that

$$B_1 \subseteq E \subseteq B_2 \subseteq [0,1), \text{ and}$$

$$m(B_1) = m(E) = m(B_2).$$

However, $S_y(B_1) \subseteq S_y(E) \subseteq S_y(B_2)$, and therefore $S_y(E)$ is Lebesgue measurable and

$$m(E) = m(S_y(E)), \text{ concluding the proof.}$$

Our next result is an analogue of 3) in section 1.

THEOREM 4. *If* $y = 0 \cdot y_1 y_2 \ldots \in [0,1)$ *and* $(n : y_n = 0)$ *and* $(n : y_n = 1)$ *are both infinite sets, then* $(S_y)'(x)$ *, the derivative of* S_y *at* x*, exists nowhere in* $[0,1)$.

P r o o f. Let $x \in [0,1)$ and let

$x = 0 \cdot x_1 x_2 x_3 \ldots$ be its binary development.

Define the sequence $(h_n)_{n=1}^{n=\infty}$ in the following way:

$$h_n = 1/2^n \text{ if } x_n = 0 \text{ and } h_n = -1/2^n \text{ if } x_n = 1.$$

A simple calculation shows that:

$$\frac{S_y(x_n + h_n) - S_y(x_n)}{h_n} = \begin{cases} 1 & \text{if } (x_n, y_n) = (0,0) \\ -1 & \text{if } (x_n, y_n) = (0,1) \\ 1 & \text{if } (x_n, y_n) = (1,0) \\ -1 & \text{if } (x_n, y_n) = (1,1) \end{cases}$$

Since the sets $(n: y_n = 0)$ and $(n: y_n = 1)$ are both assumed to be infinite it follows that the sequence

$$\left\{ \frac{S_y(x_n + h_n) - S_y(x_n)}{h_n} \right\}_{n=1}^{n=\infty}$$

contains infinitely many minus ones and infinitely many ones and therefore $(S_y)'(x)$ does not exist.

The next theorem is the switch function analogue of Theorem 3 in [1].

THEOREM 5. *If* $E \subseteq [0,1)$, $m(E) = \gamma > 0$ *and* $\varepsilon > 0$, *then there exists* y_0, $0 \le y_0 \le 1$, *such that* $0 \le y \le y_0$ *implies* $m(E \cap S_y(E)) > \gamma - \varepsilon$.

P r o o f. The proof of this theorem will not be given since it is completely analogous to the proof of Theorem 3 in [1].

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REZIME

O JEDNOJ KLASI FUNKCIJA

U radu [3] H.Steinhaus je uveo pojam permutacione funkcije intervala [0,1] i dokazao nekoliko teorema o ovim

funkcijama. A.Mookhopadhyaya [2] i H.Miller [1] takodje imaju nekoliko rezultata koji se odnose na Steinhausove permutacione funkcije. U ovom radu se definiše klasa "switch funkcija" i dokazuju za ovu klasu funkcija rezultati analogni onim koji su dobijeni za Steinhausove permutacione funkcije.

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A GENERALIZATION OF A DIEUDONNE THEOREM FOR A NONADDITIVE SET FUNCTIONS

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ABSTRACT

In this paper the famous Dieudonné theorem is generalized. If M is a family of triangular set functions defined on the family B of all Borel sets of a locally compact set T with regular variations and M is bounded on every open set, then M is uniformly bounded.

1. INTRODUCTION

As it is well-known, the Nikodym boundedness theorem for measures in general fails for algebras of sets (see Example 5., Diestel, Uhl [2], p.18). But there are uniform boundedness theorems in which the initial boundedness conditions are on some subfamilies of a given σ-algebra; those subfamilies must not be σ-algebras. A famous theorem of Dieudonné [3] states that for compact metric spaces the pointwise boundedness of a family of Borel regular measures on open sets implies its uniform boundedness on all Borel sets. We shall generalize this Dieudonné theorem on a wider class of set functions. The class ,of finitely additive regular Borel set functions gives nothing new, because each finitely additive regular Borel set function (also in the case of vector measures) is necessarily countably additive - Kupka [7].

We take in this paper a wider class of real valued set functions, the so called triangular set functions. We prove a generalized Dieudonné type theorem for this class of set functions. Using some modifications we obtain also a generalization of Dieudonné's theorem for semigroup valued set functions.

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2. TRIANGULAR SET FUNCTIONS

Let T be a locally compact space and S a class of subsets of T such that $\emptyset \in S$.

DEFINITION 1. *(Dinculeanu [4], p. 303). A set function* $\mu: S \to R$ *is said to be* <u>*regular*</u> *if for every* $A \in S$ *and every* $\varepsilon > 0$ *there exist a compact set* $K \subset A$ *and an open set* $G \supset A$ *such that for every set* $A' \in S$, $K \subset A' \subset G$, *we have*

$$|\mu(A) - \mu(A')| < \varepsilon$$

DEFINITION 2. *A set function* $\mu: S \to R$ *is said to be* <u>*triangular*</u> *if for every* $A, B \in S$, *such that* $A \cap B = \emptyset$ and $A \cup B \in S$, *we have*

$$\mu(A) - \mu(B) \leq \mu(A \cup B) < \mu(A) + \mu(B).$$

and $\mu(\emptyset) = 0$.

The following theorem is important for further characterization of set functions which are both regular and triangular.

THEOREM 1. *Let* S *be a ring of subsets of* T. *If a set function* μ $S \to R$ *is regular and superadditive, i. e.*
$\mu(A \cup B) \geq \mu(A) + \mu(B)$ for every $A, B \in S$, $A \cap B = \emptyset$
then it satisfies the following condition

(R) *For every* $A \in S$ *and every number* $\varepsilon > 0$ *there exist a compact set* $K \subset A$ *and an open set* $G \supset A$ *such that for every set* $B \in S$ *With* $B \subset G \setminus K$ *we have*

$$|\mu(B)| < \varepsilon$$

P r o o f. It is enough to adapt the proof of Proposition 1. on page 304 in [4].

COROLLARY 1. *If a set function* $\mu: S \to R$ *(S is a ring),* $\mu(\emptyset) = 0$, *has a regular variation, where the variation* $|\mu|$ *is defined in the usual way, i. e.*

$$|\mu|(E) := \sup_{\pi} \sum_{A \in \pi} |\mu(A)| \quad (E \in S)$$

and the supremum is taken over all partitions π *of* E *into a finite number of pairwise disjoint members of* S, *then* μ *satisfies the condition* (R).

P r o o f. Since $|\mu|$ is superadditive - [4] , p.34, we can apply Theorem 1. on $|\mu|$. Then the inequality $\mu \leq |\mu|$ implies our statement.

It is obvious that a triangular set function μ with a regular variation is itself regular.

DEFINITION 3. *A set function $\mu: S \to R$ is said to be exhaustive whenever given a sequence (E_n) of pairwise disjoint member of S,* $\lim_{n\to\infty} \mu(E_n)=0$.

3. UNIFORM BOUNDEDNESS THEOREM

We shall take from now on for the class S the collection B of all Borel sets of a Hausdorff locally compact topological space T. Now we shall formulate the main theorem.

THEOREM 2. *Let M be a family of triangular set functions defined on B with regular variations. If the set*

$$\{\mu(O),\ \mu \in M\}$$

is bounded for every open set O, *then*

$$\{\ \mu(B);\ \mu \in M, B \in B\}$$

is a bounded set.

REMARK 1. We shall assume in the following proofs that T is a compact Hausdorff space. Namely, we can replace T with an Alexandrov one point ω compactification $T\cup\{\omega\}$, taking $\mu(\omega)=0$ $(\mu \in M)$

We obtain easily the following

COROLLARY 2. *Let M be a family of regular scalar measures defined on B. If the set*

$$\{|\mu(O)|;\ \mu \in M\}$$

is bounded for every open set O, *then*

$$\{|\mu(B)|;\ \mu \in M,\ B \in B\}$$

is a bounded set.

P r o o f. Let $\nu(B):=|\mu(B)|$ $(B \in B, \mu \in M)$. It is obvious that the family F of all such set functions ν satisfies the conditions of Theorem 2. (by Proposition 24. from [4], p. 319 $|\nu|=|\mu|$ is also regular). So we apply Theorem 2.

In the proof of Theorem 2 we need two lemmas.

LEMMA 1. *Let μ be a triangular set function defined on B with a regular variation. Then μ is σ-subadditive on each sequence of disjoint open sets (O_n)*, i. e.

$$\mu\left(\bigcup_{j=1}^{\infty} O_j\right) \leq \sum_{j=1}^{\infty} \mu(O_j)$$

P r o o f of Lemma 1. First, we shall prove that μ is order continuous on open sets, i. e. for each sequence (U_n) of open sets such that $U_j \supset U_{j+1}$ $(j \in N)$ and $\bigcap_{j=1}^{\infty} U_j = \emptyset$ holds

$$\lim_{j\to\infty} \mu(U_j) = 0.$$

For each $\varepsilon>0$ there exists a sequence of compact sets (K_n) such that $K_j \subset U_j$ and

$$(1) \qquad |\mu|(U_j \setminus K_j) < \frac{\varepsilon}{2^j} \quad (j \in N).$$

Then there exists $n_o \in N$ such that $\bigcap_{j=1}^{n} K_j = \emptyset$ for all $n \geq n_o$. Let $n \geq n_o$. Then we have

$$\mu(U_n)=\mu(U_n \setminus \bigcap_{j=1}^{n} K_j)=\mu\left(\bigcup_{j=1}^{n} (U_n \setminus K_j)\right) \leq$$

$$\leq |\mu|\left(\bigcup_{j=1}^{n} (U_n \setminus K_j)\right).$$

Hence, since $|\mu|$ is subadditive (i. e. $|\mu|(A \cup B) \leq |\mu|(A) + |\mu|(B)$ for every papir A,B of not necessarily disjoint sets from B-analogously as in [4] , p. 35-36 and p. 16) and nondecreasing, we obtain by (1)

$$\mu(U_n) \leq \sum_{j=1}^{n} |\mu|(U_j \setminus K_j) < \varepsilon$$

for all $n \geq n_o$. Now, let (O_n) be a sequence of disjoint open sets. Then we have

$$\mu\Big(\bigcup_{j=1}^{\infty} O_j\Big) \leq \sum_{j=1}^{\mu} \mu(O_j) + \mu\Big(\bigcup_{j=n+1}^{\infty} O_j\Big).$$

Taking $n \to \infty$ we obtain

$$\mu\Big(\bigcup_{j=1}^{\infty} O_j\Big) \leq \sum_{j=1}^{\infty} \mu(O_j).$$

The following lemma is given by C. Swartz in [12] as an extract from the elementary proof of the Antosik-Mikusiński diagonal theorem - [1].

LEMMA 2. *Let* X *be a Banach space. If* $x_{ij} \in X$ $(i,j \in N)$ *such that* $\lim_{j\to\infty} x_{ij} = 0$ $(i \in N)$, $\lim_{i\to\infty} x_{ij} = 0$ $(j \in N)$ and $\|x_{ii}\| \geq \delta > 0$ $(i \in N)$, *then there exist a sequence* (i_n) *of natural numbers and a sequence* (ε_n) *of positive real numbers such that*

$$\Big\| \sum_{k=1}^{n-1} x_{i_n i_k} \Big\| = \Big(\frac{1}{2} - \varepsilon_n\Big) \|x_{i_n i_n}\|; \ \|x_{i_n i_{n+g}}\| < 2^{-2}\varepsilon_n \|x_{i_n i_n}\|$$

(in [12] is δ instead of $\|x_{i_n i_n}\|$).

Proof of Theorem 2. It suffices to prove that every point in T belongs to an open set O so that

(2) $$\sup\{\mu(A) : A \subset O (A \in \mathcal{B}), \mu \in \mathcal{M}\} < \infty.$$

Suppose that this is not true. Then there exists a point $x \in T$ such that (2) does not hold for every open set O such that $x \in O$. We shall prove that there exists a sequence of pairwise disjoint open sets (E_n) and a sequence (μ_n) from $\mathcal{M}$ such that

$$\mu_i(E_i) > i \quad (i \in N).$$

For any open set O such that $x \in O$ there exists a Borel set $B \subset O$ and $\mu_1 \in \mathcal{M}$ such that

(3) $$\mu_1(B) > 4 + 2 \sup_{\mu \in \mathcal{M}} \mu(\{x\}).$$

It is easy to prove that the preceding supremum is finite. Since

μ_1 has a regular variation by Corollary 1 there exists a compact set $K \subset B$ and an open set $O' \subset O$, $B \subset O'$ such that

$$\mu_1(B') < 1$$

for each $B' \subset O' \setminus K$. We have by the subadditivity of μ_1

$$\mu_1(K) + \mu_1(B \setminus K) \geq \mu_1(B).$$

Using the preceding inequality, the inequality

$$\mu_1(B \setminus K) < 1$$

and (3) we obatain

$$\mu_1(K) > 3 + 2 \sup_{\mu \in M} \mu(\{x\}).$$

Let $K_1 = K \cup \{x\}$. Then the last inequality implies (directly for $x \in K$) by the triangularity of μ_1 (for $x \notin K$)

$$\mu_1(K_1) > 3 + \sup_{\mu \in M} \mu(\{x\}).$$

By the regularity of μ_1 there exists an open set U such that $O \supset U \supset K_1$ and

$$\mu_1(B') < 1 \text{ for every } B'' \subset U \setminus K_1.$$

The preceding inequality together with the inequality

$$\mu_1(U) \geq \mu_1(K_1) - \mu_1(U \setminus K_1)$$

implies

(4) $$\mu_1(U) > 2 + \sup_{\mu \in M} \mu(\{x\}).$$

Again by the reqularity of μ_1 there exists an open set W such that $\{x\} \subset W \subset U$ and

(5) $$\mu_1(B''') < 1$$

for every $B''' \subset W \setminus \{x\}$.

Let H be an open set such that $x \in H \subset \bar{H} \subset W$ (where $\bar{H}$ is the closure of the set H). Then we have

$$\mu_1(\bar{H}) \leq \sup_{A \subset \bar{H} \setminus \{x\}} \mu_1(A) + \mu_1(\{x\})$$

$$\leq \sup_{B \subset W \setminus \{x\}} \mu_1(B) + \mu_1(\{x\}).$$

Hence by (5) we obtain

(6) $$\mu_1(\bar{H}) < 1 + \sup_{\mu \in M} \mu(\{x\}).$$

Let $E_1 = U \setminus \bar{H}$. Then we have $E_1 \subset O$ and $E_1 \cap \bar{H} = \emptyset$

By the inequality

$$\mu_1(E_1) + \mu_1(\bar{H}) \geq \mu_1(U) .$$

(4) and (6) we obtain

$$\mu_1(E_1) > 1.$$

Using the preceding procedure, taking in inequality (3)

" $5 + 2 \sup_{\mu \in M} \mu(\{x\})$ " instead of " $4 + 2 \sup_{\mu \in M} \mu(\{x\})$"

and taking into account the facts that: $x \in H$ and the family M is not bounded on H, we obtain the open sets E_2, H_1 $(H_1 \subset H)$ and $\mu_2 \in M$ such that $E_2 \cap H_1 = \emptyset$, $x \in H_1$ and

$\mu_2(E_2) > 2$. We have $E_1 \cap E_2 = \emptyset$.

Continuing this procedure we obtain a sequence (μ_i) from M and a sequence (E_i) of pairwise disjoint open sets such that

(7) $$\mu_i(E_i) > i \quad (i \in N).$$

We shall prove that μ_i $(i \in N)$ are exhaustive on a sequence (E_n) of disjoint open sets, i. e.

(8) $$\lim_{j \to \infty} \mu_i(E_j) = 0 \quad (i \in N).$$

Since $\bigcup_{j=1}^{\infty} E_j$ is an open set and $|\mu_i|$ are regular, for $\varepsilon > 0$ by Corollary 1 there exists a compact set $K \subset \bigcup_{j=1}^{\infty} E_j$ such that

$\mu_i(C) < \varepsilon$ for each $i \in N$ and each

$C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$. Since (E_i) is an open cover of K' so there exists $n_o \in N$ such that $K' \subset \bigcup_{j=1}^{n_o} E_j$.

Then we have for $m \geq n_o + 1$

$$\mu_i(E_m) \leq \sup_{C'} \mu_i(C') \leq \sup_{C} \mu_i(C) < \varepsilon \; (i \in N)$$

where $C' \subset E_m \cup \left(\bigcup_{j=1}^{n_o} E_j \setminus K' \right)$ and $C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$.

So we obtain (8).

Let $x_{ij} = \mu_i(E_j)/i$. We have by (8) $\lim_{j\to\infty} x_{ij} = 0$ $(i \in N)$.

We obatin by the boundedness assumption of the theorem $\lim_{i\to\infty} x_{ij} = 0$ $(j \in N)$. Applying Lemma 2 on the infinite matrix $|x_{ij}|$ $(i,j \in N)$ we obtain a sequence (i_n) from N and a sequence (ε_n) of positive real numbers such that

$$\sum_{k=1}^{n-1} x_{i_n i_k} = \left(\frac{1}{2} - \varepsilon_n \right) x_{i_n i_n} \tag{9}$$

$$x_{i_n i_{n+q}} < 2^{-q} \varepsilon_n \, x_{i_n i_n} \quad (n \in N). \tag{10}$$

Using the triangularity of μ_{i_n} $(n \in N)$ and Lemma 1 we obtain

$$\mu_{i_n}\left(\bigcup_{k=1}^{\infty} E_{i_k} \right) \geq \mu_{i_n}(E_{i_n}) - \sum_{k=1}^{n-1} \mu_{i_n}(E_{i_k}) - \sum_{k=n+1}^{\infty} \mu_{i_n}(E_{i_k})$$

$(n \in N)$. Hence by (9) and (10)

$$i_n^{-1} \mu_{i_n}\left(\bigcup_{k=1}^{\infty} E_{i_k} \right) \geq x_{i_n i_n} - \sum_{k=1}^{n-1} x_{i_n i_k} - \sum_{k=n+1}^{\infty} x_{i_n i_k} \geq$$

$$\geq \frac{x_{i_n i_n}}{2} \quad (n \in N), \text{ i. e.}$$

$$\mu_{i_n}\left(\bigcup_{k=1}^{\infty} E_{i_k} \right) \geq \frac{\mu_{i_n}(E_{i_n})}{2} \quad (n \in N).$$

Then by (7) we obtain

$$\mu_{i_n}\left(\bigcup_{k=1}^{\infty} E_{i_k}\right) \geq \frac{i_n}{2} \quad \text{for each } n \in N.$$

Since $\bigcup_{k=1}^{\infty} E_{i_k}$ is an open set we obtain a contradiction with the boundedness of (μ_{i_n}) on open sets.

4. FURTHER GENERALIZATIONS

Let X be a commutative semigroup with a neutral element O. Let $d:X \to [0,+\infty)$ be a pseudometric which satisfies the following condition

$$(d_+) \quad d(x+x_1, y+y_1) \leq d(x,y) + d(x_1,y_1)$$

for all $x, x_1, y, y_1 \in X$.

EXAMPLE. Weber [13] has proved that for every commutative complete uniform semigroup there exists a family of pseudometrics which satisfy (d_+) and which generate its uniformity.

Let X be endowed with a pseudometric d which satisfies (d_+). Now we can extend the definition of the regularity of a set function $\nu: S \to X$ only taking in Definition 1 ν and "$d(\nu(A), \nu(A')) < \varepsilon$" instead of μ and "$|\nu(A) - \nu(A')| < \varepsilon$" respectively.

The pseudometric d induces a triangular functional - E. Pap [8], [10] in the following way

$$f(x) := d(x, O) \quad (x \in X).$$

The functional f satisfies

$$(F_1) \quad f(x+y) \leq f(x) + f(y) \quad \text{and}$$

$$(F_2) \quad f(x+y) \geq |f(x) - f(y)| \quad \text{for all } x, y \in X.$$

Now we define the variation $|\nu|$ of a set function $\nu: S \to X$ with $\nu(\emptyset) = 0$ in the following way

$$|\nu|(E) := \sup_{\pi} \sum_{A \in \pi} f(\nu(A)) \qquad (E \in S)$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of S. It is easy to

see that $|\nu|$ is superadditive.

A set function $\nu: \mathcal{S} \to X$ is said to be a semigroup valued triangular set function if it satisfies

$$\nu(\emptyset) = 0,$$

$$f(\nu(A)) - f(\nu(B)) \leq f(\nu(A \cup B)) \leq f(\nu(A)) + f(\nu(B))$$

for $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$.

Now we have the following generalization of Theorem 2.

THEOREM 3. *Let F be a family of semigroup valued triangular set functions with regular variations defined on $\mathcal{B}$. If the set*

$$\{ f(\nu(O)); \nu \in F\}$$

is bounded for every open set O, then

$$\{ f(\nu(B)); \nu \in F,\ B \in \mathcal{B}\}$$

is a bounded set.

P r o o f. We take $\mu(B) := f(\nu(B)) \quad (B \in \mathcal{B}, \nu \in F)$ and we apply Theorem 2.

REMARK 2. Theorem 3. holds also for a family of N-triangular set functions $\nu: \mathcal{B} \to G$ ($(G, |\ |)$ is a quasinormed group) with a constant $N \in (0, \infty)$ - [6], [4], i. e. such that $\nu(\emptyset) = 0$ and

$$|\nu(A)| - N\,|\nu(B)| \leq |\nu(A \cup B)| \leq |\nu(A)| + N|\nu(B)|$$

for all disjoint $A, B \in \mathcal{B}$.

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REZIME

JEDNO UOPŠTENJE TEOREME DIEUDONNEA NA NEADITIVNE SKUPOVE FUNKCIJE

U radu se kao uopštenje teoreme Dieudonné-a dokazuje teorema o uniformnoj ograničenosti familije, u opštem slučaju, neaditivnih skupovinih funkcija. Klasa izučavanih skupovnih funkcija se sastoji od tkzv. trougaonih skupovnih funkcija. $\mu:S\to R$ (S je familija podskupova lokalno kompaktnog prostora i $\emptyset \in S$ je trougaona skupovna funkcija ako za svako $A, B\in S$, tako da je $A\cap B=\emptyset$ i $A\cup B \in S$, važi

$$\mu(A) - \mu(B)\leq\mu(A\cup B)\leq\mu(A) + \mu(B) \quad \text{i} \quad \mu(\emptyset) = 0.$$

Neka je M familija trougaonih skupovnih funkcija definisanih na familiji B svih Borelovih podskupova lokalno kompaktnog prostora T sa regularnim varijacijama. Ako je familija M ograničena nad svakim otvorenim skupom, tada je ona i uniformno ograničena (teorema 2). Na kraju se dobijeni rezultat prenosi i na skupovne funkcije sa vrednostima u komutativnoj polugrupi i na N-trougaone skupovne funkcije sa vrednostima u komutativnoj grupi sa kvazi-normom [6], [4].

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A SIMPLE PROOF OF A GENERALIZED DIEUDONNÉ THEOREM

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ABSTRACT

A simple proof is given of a Dieudonné type theorem on uniform boundedness of a family of regular measures defined on the family of all Borel sets of an arbitrary Hausdorff topological space.

A famous theorem of Dieudonné states that for a compact metric space T the pointwise boundedness of a family of Borel regular measures on the family of all open sets of T implies its uniform boundedness on the family of all Borel sets of T. There are several generalizations of this theorem [2], [5], [7], [8]. A funitely additive regular Borel set function reduces to a measure even for a Banach space valued set function, as wes pointed out by J. Kupka [5], the proof is given in [3]. In [7] T is a regular (T_3) space and in [2] T is a locally compact space. In this note we shall give an easy proof of a Dieudonne type theorem for the case when T is an arbitrary

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Hausdorff topological space. The proof is quite elementary and short. It is based on two lemmas which are connected with certain properties of numbers.

LEMMA 1. (P. Antosik [1].) *Let* x_{ij} $(i,j \in N)$ *be complex numbers. If* $\lim_{j\to\infty} x_{ij} = 0$ *for* $i = 1,2,\ldots,$ *then there exist an infinite set* I *of positive integers and a subset* J *(finite or infinite) of* I *such that for all* $i \in I$ *we have*

$$\sum_{j\in J} |x_{ij}| < \infty$$

and

$$\left|\sum_{j\in J} x_{ij}\right| \geq \frac{1}{2}\, |x_{ii}|.$$

REMARK 1. Originally Lemma 1 was stated in [1] for elements from normed space.

Let T be a Hausdorff topological space and B the collection of all Borel subsets of T. A Borel measure μ on T is regular: if B is a Borel subset of $T, \varepsilon > 0$, then there exists a compact subset $K \subset B$ with $|\mu|(B\setminus K) < \varepsilon$.

LEMMA 2 . (J.D.Stein [7], Lemma 1.). *If* μ *is a nonzero regular measure on* T *and* U *is an open subset of* T*, then the following holds:*

There exists an open set V *with* $V \subset U$ *and*

$$|\mu(V)| > |\mu|(U)/7 .$$

The easy proof of this Lemma in [7] is based on an inequality for complex number [6]: if $z_1, z_2, \ldots, z_n$ are complex numbers, there is a subset S of $\{1,2,\ldots,n\}$ such that

$$\left|\sum_{j\in S} z_j\right| \geq \frac{1}{6} \sum_{j=1}^{n} |z_j| .$$

REMARK 2. Lemma 2 is stated in [7] for the case when T is a regular (T_3) topological space but the same proof holds also when T is a Hausdorff topological space.

THEOREM. *Let* M *be a family of regular scalar measures defineb on* B. *If the set*

$$\{|\mu(O)| \; |\mu \in M\}$$

is bounded for every open set O, *then*

$$\{|\mu(B)| \; |\mu \in M, \; B \in B\}$$

is a bounded set.

P r o o f. Firstly, let us suppose that the set of variations of the elements of the family M is unbounded on an open set O. Then for some $M_1 > 0 |\mu|(O) < M_1$ $(\mu \in M)$ and for each $M > 0$ and each $\varepsilon > 0$ there exists $\mu \in M$ such that $|\mu|(O) > 7(M+M_1+3\varepsilon)$. By Lemma 2 there exists an open set $W \subset O$ such that $|\mu(W)| > M+M_1+3\varepsilon$. Since μ is regular there exists a compact subset K_1 of W such that $|\mu(W \setminus K_1)| < \varepsilon$. Hence $|\mu(K_1)| > M+M_1+2\varepsilon$. Again by the regularity of μ there exists a compact subset K_2 of $O \setminus K_1$ such that $|\mu|((O \setminus K_1) \setminus K_2) < \varepsilon$, thus $|\mu(K_2)| > M+\varepsilon$. Since $K_1 \cap K_2 = \emptyset$ and T is a Hausdorff topological space there exist disjoint open sets H and V such that $H \supset K_1$ and $V \supset K_2$ such that $|\mu(V \setminus K_2)| < \varepsilon$ and $|\mu(H \setminus K_1)| < \varepsilon$. Then we obtain

$$|\mu(V)| > M \quad \text{and} \quad |\mu(H)| > M .$$

Let us assume that the theorem is not true, i.e.

$$\nu(T) = \infty \;, \quad \text{where} \quad \nu(B) = \sup_{\mu \in M} |\mu|(B) \quad (B \in B).$$

By the preceeding for any given arbitrary large positive number there exist $\mu_1 \in M$ and two disjoint open subsets H_1 and V_1 of T, such that $|\mu_1(H_1)|$ and $|\mu_1(V_1)|$ are

greater than it. Then there are two possibilities. Either one of $\nu(H_1)$ or $\nu(V_1)$ is infinite (in this case we take $\nu(H_1) = \infty$) or both are finite.

In the first case, we apply the preceding procedure on H_1 (instead of T). Then for any given positive number, we can find a measure $\mu_2 \in M$ and disjoint open sets H_2 and V_2 such that $|\mu_2(H_2)|$ and $|\mu_2(V_2)|$ are greater than it. Thus, subsequently, by repeating the process, supposing always that $\nu(H_i) = \infty$, we can obtain a sequence (μ_i) from M and a sequence of disjoint open sets (V_i) such that

$$(1) \qquad |\mu_i(V_i)| > i \quad (i \in N) .$$

Because of $\lim_{j\to\infty} \mu_n(V_i) = 0$ $(n \in N)$ we can apply Lemma 1 on $\mu_i(V_j)$ $(i,j \in N)$. According to this there exist an infinite set $I \subset N$ and its subset J such that for each $i \in I$

$$\Big|\sum_{j\in J} \mu_i(V_j)\Big| \geq \frac{1}{2}\,|\mu_i(V_i)| .$$

By the preceding inequalities and (1), we obtain $|\mu_i(V)| > i$ $(i \in I)$ for the open set $V = \bigcup_{j\in J} V_j$, contradictory to the condition of the boundedness of the family M on open sets.

Suppose now that both $\nu(H_1)$ and $\nu(V_1)$ are finite. This implies $\nu((T\setminus K_1)\setminus K_2)) = \infty$. Now we can apply the preceding procedure on the open set $T\setminus(K_1 \cup K_2)$ (instead of T). Thus, sequently, by repeating the process, supposing always that $\nu(H_i)$ and $\nu(V_i)$ are finite (i.e. there exists $p_i \in N$ such $\nu(H_i) < p_i$ and $\nu(V_i) < p_i$) we can obtain a sequence (μ_i) from M and a sequence of open sets (in general not disjoint) (V_i) such that

$$(2) \qquad |\mu_i(V_i)| > i + c_{i-1} + \varepsilon \qquad (i \in N) ,$$

where $c_i = \sum_{k=1}^{i} p_k$.

By
$$\nu\Big(\bigcup_{k=1}^{i-1} V_k \setminus V_i\Big) \le c_{i-1} \qquad (i \in N) \quad ,$$

$$|\mu_i|\Big(\bigcup_{k=i+1}^{\infty} V_k \setminus \bigcup_{k=1}^{i} V_k\Big) < \varepsilon \qquad (i \in N)$$

and (2) we obtain

$$|\mu_i(V)| > i \qquad (i \in N)$$

for the open set $V = \bigcup_{i=1}^{\infty} V_i$. Again a contradiction.

Finally, we reduce the general case to the preseding two. Namely, in general, we combine the preceding two procedures, always taking, as the initial set for the next step, that the open set for which ν is infinite.

If there is an infinite set $I \subset N$ such that $\nu(H_i) = \infty$ $(i \in I)$, then, passing to a subsequence we reduce all to the first case. In the opposite case, we reduce all to the second case.

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REZIME

JEDAN JEDNOSTAVAN DOKAZ UOPŠTENJA DIEUDONNE-OVE TEOREME

U radu je dat jednostavan elementaran dokaz uopštenja Dieudonne-ove teoreme o uniformnoj ograničenosti familije regularnih mera. Neka je M familija regularnih skalarnih mera definisanih nad famijom $\mathcal{B}$ svih Borelovih podskupova Hausdorffovog prostora T. Ako je skup

$$\{|\mu(O)| \ |\mu \in M\}$$

ograničen za svaki otvoren skup O, tada je

$$\{|\mu(B)| \ |\mu \in M, B \in \mathcal{B}\}$$

ograničen skup.

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A NOTE ON SOME CONVERGENCES ON SEMIGROUPS

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ABSTRACT

In the paper a new notion on a commutative semigroup with a nontrivial subadditive and homogeneous functional - Chauchysequence condition is introduced and with it a general theorem on convergences is obtained. As consequences of this theorem, some Orlicz-Pettis theorems are obtained.

1. INTRODUCTION

The point in the proofs of many Orlicz-Pettis type theorems ([2], [8], [9]) is: if $\Sigma_n x_n$ is weak subseries convergent then the sequence (x_n) has a subsequence which is norm convergent to 0. The purpose of this paper is to prove a general theorem - Theorem 3.3, on a commutative semigroup which extracts this connection between convergences. Let us observe that even in the classical case the approach is a new one.

An important tool is a generalization of the Hahn-Banach theorem on commutative semigroups ([3], [4]) which gives us the existence of a nontrivial special additive functional.

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2. SUBADDITIVE, HOMOGENEOUS AND ADDITIVE FUNCTIONALS

Let X be a commutative semigroup. A functional $f:X \to R_+$ (R_+ is the set of all nonnegative real numbers) will be called a <u>subadditive</u> functional if it satisfies the following condition

$$(F_1) \quad f(x+y) \le f(x) + f(y) \quad \text{for all } x,y \in X.$$

REMARK 1. H.Weber [10] has proved that for every commutative complete uniform semigrup there exists a family of pseudometrics d which satisfy $d(x+x_1, y+y_1) \le d(x,y)+d(x_1,y_1)$ for all $x, x_1, y, y_1 \in X$ and which generate its uniformity. Such a pseudometric induces a subadditive functional f in the following way

$$f(x) := d(x,0)\,(x \in X).$$

We say that a functional $f:X \to R_+$ if <u>homogeneous</u> if

$$(F_2) \quad f(nx) = n\,f(x) \quad (x \in X,\ n \in N).$$

The condition (F_2) is independent from (F_1). For example, let (h_k) be a Hamel basis for a vector space, then for $x = \Sigma a_k h_k \in X$ we define $p(x) = \Sigma\sqrt{|a_k|}$. Obviously $p(.)$ is a quasi-norm, but $p(nx) = \sqrt{n}\,p(x)$ for all $n \in N$ and all $x \in X$.

To each subadditive functional we can correspond a homogeneous functional which is closely connected with the original one in the following way.

PROPOSITION 2.1. *Let f be a subadditive functional on a commutative semigroup X. Then there exists a homogeneous functional F on X such that*

(i) F *is subadditive,*

(ii) $F(x) \le f(x) \quad (x \in X)$:

P r o o f. We take that

$$F(x) = \inf\{\tfrac{1}{n} f(nx) \mid n \in N\} \quad (x \in X).$$

As an easy consequence of the generalized Hahn-Banach theorem from [3] and [4] (also in [7]) we can obtain the following theorem.

THEOREM 2.2. *Let* X *be a commutative semigroup and* f *be a homogeneous finite subadditive functional on* X. *If* x_o *is an element from* X *such that* $f(x_o)\neq 0$, *then there exists an additive functional* h *on* X *such that* $h(x_o) = f(x_o)$ *and* $h(x) \leq f(x)$ *for all* $x \in X$.

REMARK 2. Condition $f(x_o) \neq 0$ from the preceding theorem implies $nx_o \neq x_o$ for each $n \in N$, i.e. x_o is not of a finite order.

3. MAIN RESULTS

Let X be a commutative semigroup with a neutral element O and with a nontrivial homogeneous subadditive functional f.

The following notion will be crucial in the main theorem 3.3 of this section. Let (y_j) be a sequence of elements from X and H be a family of additive functionals defined on X such that $h(x) \leq f(x)$ $(x \in X)$. Then a subset X_1 of X will be called a $((y_j),H)$ - subsemigroup if: x belongs to X_1 iff $h(u_1+ +\ldots+u_k) \to h(x)$ as $k \to \infty$ and all $h \in H$ for some sequence (u_j) such that u_j is either $\lambda_j y_j$ (for $\lambda_j \in N$) or O and $\sum_{j=1}^{\infty} |h(u_j)| < \infty$ $(h \in H)$.

X_1 is nonempty. Namely, O and all the members and the finite sums of the members of the sequence (y_j) belong to X_1. Since the series $\sum_{j=1}^{\infty} h(u_j)$ $(h \in H)$ are unconditionally convergent it is easy to see that X_1 is really a subsemigroup of X.

We need in the proofs of Theorem 3.3 and Theorem 3.4 the following theorem. We always have finite additive functionals.

THEOREM 3.1. *Let* (h_n) *be a sequence of additive functionals on a commutative semigroup* X. *Let* (x_n) *be a sequence from* X *such that for its every subsequence* (z_n) *there exist a subsequence* (y_n) *of* (z_n) *and an element* y *from* X *such that*

$$h_n(y_1 + \ldots + y_k) \to h_n(y)$$

as $k \to \infty$ *for each* $n \in N$.

Then there exist an infinite set $I \subset N$ *and an element* x *from* X *such that for all* $n \in I$

$$\sum_{j\in J} |h_n(x_j)| < \infty \quad \textit{for some } J \subset I,$$

$$|h_n(x)| \geq \frac{1}{2} |h_n(x_n)| .$$

Since $h_n(\cdot)$ are triangular functionals (i.e. subadditive and $|h_n(x+y)| \geq |h_n(x)| - |h_n(y)|$ for $x, y \in X$, $|h_n(0)| = 0$) the proof of Theorem 3.1 is analogous to the proof of the Antosik-Mikusiński Diagonal Theorem [1], [5] and [6] (using in the second part of the proof the assumption on sequence (x_n)).

Let H be a family of finite additive functionals h on X with the property $h(x) \leq f(x)$ $(x \in X)$. We say that X satisfies the H-Cauchy sequence condition if for each sequence (y_j) from X such that

$$\sum_{j=1}^{\infty} | h(y_j) | < \infty \quad (h \in H)$$

and each sequence (h_n) from H such that it is a Cauchy sequence on (y_j), i.e. for each $\varepsilon > 0$ there exists $n_o \in N$ such that

$$|h_n(y_j) - h_m(y_j)| < \varepsilon \quad \text{for all} \quad n, m \geq n_o = n_o(j),\ j \in N,$$ then (h_n) is a Cauchy sequence on the $((y_j), H)$ - subsemigroup.

It is easy to see that if X is a finite semigroup or H is a finite family, then X satisfies the H-Cauchy sequence condition.

In a specially important case we have the following proposition.

PROPOSITION 3.2. *Each normed space X satisfies the B^* - Cauchy sequence condition (B^* is the unit ball in the dual X^*).*

P r o o f. We shall prove that each sequence (h_n) of continuous linear functionals from B^* which is a Cauchy sequence on each member of the sequence (y_j) is a Cauchy sequence on the whole closed linear subspace $\overline{L((y_j))}$ generated by (y_j).

Let $x \in \overline{L((y_j))}$. Then for each $\varepsilon > 0$ there exist $\lambda_1, \dots$ $\dots, \lambda_{k_o}$ such that

$$\| \lambda_1 y_1 + \dots + \lambda_{k_o} y_{k_o} - x \| < \frac{\varepsilon}{3}$$

Since (h_n) is a Cauchy sequence on (y_n) there exists $n_o \in N$ such that

$$|h_m(y_j) - h_n(y_j)| < \frac{\varepsilon}{3|\lambda_j| k_o} \quad (\lambda_j \neq 0)$$

for each $n,m \geq n_o$ and each $j=1,\dots,k_o$. Hence we have

$$|h_m(x) - h_n(x)| \leq 2 \| \lambda_1 y_1 + \dots + \lambda_{k_o} y_{k_o} - x \| +$$

$$+ \sum_{j=1}^{k_o} |\lambda_j| \, |h_m(y_j) - h_n(y_j)| < \varepsilon$$

for each $n,m \geq n_o = n_o(\varepsilon, x)$.

We obtain as a consequence of the Hahn-Banach theorem on normed spaces (i.e. if $s_n \in L((y_j))$ $(n \in N)$ and $h_n(s_n) \to h(s)$ as $n \to \infty$ for each $h \in B^*$, then $s \in \overline{L((y_j))}$)

$$((y_j), B^*) \subset \overline{L((y_j))} \quad .$$

We say that a family H of additive functionals on X with the property $h(x) \leq f(x)$ $(x \in X,\ h \in H)$ satisfies the <u>ε-condition</u> if for arbitrary $x_o \in X$, each $\varepsilon > 0$ and each additive functional h' on X with the property $h'(x) \leq f(x)$ $(x \in X)$ there exists $h \in H$ such that $h(x_o) + \varepsilon > h'(x_o)$.

If H is the family of all additive functionals with the property $h(x) \leq f(x)$ $(x \in X,\ h \in H)$ then it satisfies trivially the ε-condition.

E.Thomas has introduced in Theorem II.3 from [9] a subfamily H of the dual X^* of a normed space such that $\|x\| = \sup_{x^* \in H \cap B^*} |\langle x, x^* \rangle|$ $(x \in X)$. It is easy to see that such a family satisfies the ε-condition.

Now we have the main theorem.

THEOREM 3.3. *Let* X *be a commutative semigroup with a neutral element* 0 *and with a nontrivial finite homogeneous subadditive functional* f. *Let* H *be a family of additive functionals on* X *which satisfies the* ε*-condition. If* X *satisfies the* H-*Cauchy sequence condition and* (x_n) *is a sequence from* X *such that for every subsequence* (y_n) *of* (x_n) *there exists an element* $y \in X$ *such that*

$$h(y_1 + \ldots + y_n) \to h(y) \quad \textit{as } n \to \infty$$

for each $h \in H$, *then* $f(x_n) \to 0$ *as* $n \to \infty$.

P r o o f. of Theorem 3.3.

Suppose that the theorem is not true. Then for every $\varepsilon > 0$ there exists a subsequence (z_n) of (x_n) such that $f(z_n) > 4\varepsilon$ $(n \in N)$. By Theorem 2.2 there exists a sequence (h_n') of additive functionals on X such that

$$h_n'(x) \leq f(x) \quad (x \in X,\ n \in N) \text{ and } h_n'(z_n) > 4\varepsilon .$$

Since H satisfies the ε-condition we have a sequence (h_n) from H such that $h_n(z_n) > 3\varepsilon$ $(n \in N)$. We have

$$h_n(y_1 + \ldots + y_k) \to h_n(y)$$

as $k \to \infty$ $(n \in N)$ for a subsequence (y_n) of (z_n) and $y \in X$. Hence $h_n(y_j) \to 0$ as $j \to \infty$ for each $n \in N$. Then there exists a sequence (j_s) of natural numbers such that

$$(1) \qquad |h_{j_s}(y_{j_{s+q}})| < 2^{-1-q'-q} \qquad (s, q \in N),$$

where q' is a fixed natural number such that $2^{-q'} < \varepsilon$.

Now by the diagonal procedure we shall construct a subsequence of (h_{j_s}) which we denote with (q_n), such the sequence $(g_n(y_{j_s}))$ is convergent for each fix $s \in N$.

Since (x_n) is a sequence from X such that for every subsequence (u_n) of (x_n) there exists an element $u \in X$ such that

$$\sum_{n=1}^{\infty} h(u_n) = h(u) \quad \text{for each } h \in H$$

so we obtain by Riemann's theorem on convergences of series of real numbers

$$\sum_{s=1}^{\infty} |h(y_{j_s})| < \infty \quad (h \in H).$$

X satisfies the H-Cauchy sequence condition so (g_n) is a Cauchy sequence on the $((y_{j_s}),H)$ -semigroup X_1.

Now we take $g_{j_{k+1}} - g_{j_k}$ $(k \in N)$. Then by Theorem 3.1 there exist $x \in X_1$ and an infinite set $I \subset N$ such that

$$|g_{j_{k+1}}(x) - g_{j_k}(x)| \geq \frac{1}{2} |g_{j_{k+1}}(y_{j_{k+1}}) - g_{j_k}(y_{j_{k+1}})|$$

for each $k \in I$. By $g_{j_{k+1}}(y_{j_{k+1}}) > 3\varepsilon$ and (1) we obtain

$$|g_{j_{k+1}}(x) - g_{j_k}(x)| > \varepsilon$$

for each $k \in I$. A contradiction with the fact that (g_{j_k}) is a Cauchy sequence on X_1. So $f(x_n) \to 0$.

In a specially important case, when X is a normed space, we obtain by Proposition 3.2 and Theorem 3.3 the classical Orlicz-Pettis Theorem and also the Orlicz-Pettis type theorems II.3 and II.4 from [9].

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REZIME

JEDNA BELEŠKA O NEKIM KONVERGENCIJAMA NAD POLUGRUPAMA

U radu se uvodi novi pojam H-Košijev nizovni uslov, te se pomoću njega dokazuje jedna opšta teorema o konvergenciji u komutativnoj polugrupi. Neka je X komutativna polugrupa sa neutralnim elementom koja je snabdevena netrivijalnom subaditivnom i homogenom funkcionelom f. Neka je H familija

konačnih aditivnih funkcionela h nad X sa osobinom $h(x) \leq f(x)$ $(x \in X,\ h \in H)$. Kažemo da X zadovoljava H-Košijev nizovni uslov ako za svaki niz (y_j) iz X takav da je

$$\sum_{j=1}^{\infty} |h(y_j)| < \infty \quad (h \in H)$$

i svaki niz (h_n) iz H takav da je Košijev niz nad (y_j), tada je (h_n) Košijev niz i nad $((y_j),H)$-polugrupom X_1 $(x \in X_1$ ako i samo ako $h(u_1+\ldots+u_k) \to h(x)$ za $k \to \infty$ i sve $h \in H$ za neki niz (u_j) takav da je u_j ili $\lambda_j y_j$ (za $\lambda_j \in N$) ili 0 i $\sum_{j=1}^{\infty} |h(u_j)| < \infty$ $(h \in H))$.

Ako je X konačna polugrupa ili je H konačna familija tada X uvek zadovoljava H-Košijev nizovni uslov. U slučaju normiranog vektorskog prostora X, X zadovoljava B^*-Košijev nizovni uslov (B^* je jedinična lopta u dualu X^*)- Propozicija 2.1.

Za familiju H se kaže da zadovoljava ε-uslov ako za svako $x_o \in X$, svako $\varepsilon > 0$ i svaku aditivnu funkcionelu h' nad X sa osobinom $h(x) \leq f(x)$ $(x \in X)$ postoji $h \in H$ tako da je

$$h(x_o) + \varepsilon > h'(x_o).$$

U glavnoj teoremi 3.3 se dokazuje da ako niz (x_n) iz polugrupe X, koja zadovoljava H-Košijev nizovni uslov za familiju H koja zadovoljava ε-uslov, ima osobinu da za svaki njegov podniz (y_n) postoji $y \in X$ tako da je

$$h(y_1+\ldots+y_n) \to h(y) \quad \text{kada } n \to \infty \quad (h \in H),$$

tada $f(x_n) \to 0$ za $n \to \infty$.

Pomoću ove teoreme se dokazuju neke teoreme tipa Orlicz-Pettisa.

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CORRECTIONS TO "SOLVABILITY OF CONVOLUTION EQUATIONS IN $H'\{M_p\}$"

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Our paper "Solvability of Convolution Equations in $H'\{M_p\}$" was published in this Journal, Volume 11(1981),45-58.

In the proof of Theorem 5 we overlooked that β (p. 55) must not be bounded. So from this point up to the end of the paper we have to suppose the following additional assumption

(B) For every $p \in N$ there exist $p' \in N$, $\delta > 0$, and $x_\delta > 0$ such that

$$M_p^*(x) \geq M_{p'}^*(x^{1+\delta}) \quad \text{if} \quad x > x_\delta$$

To avoid missunderstendings we shall reformulate Theorem 5 and give the complete proof of it.

THEOREM 5. *Let* $F(\xi)$ *be an entire analytic function which is* M_q*-slowly decreasing for some* $q \in N$ *and let* $p \geq q'$ *where* q' *correspond to* q *in* (B). *If* $F(\xi)$ *satisfies an estimate* (9) *for some* c, n, *and this* p *then* $F(\xi)$ *is extremely slowly decreasing.*

P r o o f. We shall use the idea of the proof of Theorem 3′ from [4].

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There exists $L_1 > 0$ such that

$$\sup\{M_p^*(x)/M_q^*(x/A_1), \quad |x| \geq L_1\} \leq 1$$

holds (A_1 is from (13)). Namely if $0 < \delta_1 < \delta$ from (B) follows

$$M_q^*(x/A_1) \geq M_p^*\left(\left(\frac{x}{A_1}\right)^{1+\delta}\right) \geq M_p^*(x^{1+\delta_1}) \geq M_p^*(x)$$

for sufficiently large $|x|$.

Let us take $L \geq L_1$ so large that $\rho_1(\log(1+|\xi|)) > 1$ for each ξ with $|\xi| > L$. Let us fix ξ with $|\xi| > L$ and define

$$\beta := \frac{\log \rho}{\log(M_p^{*-1}(M_q^*(\rho/A_1))) - \log \rho}$$

where $\rho = \rho_1(\log(1+|\xi|)) > 1$. Observe that from (B) follows that $0 < \beta \leq \delta$. Let us put $\bar{R} := \rho^{(\beta+1)/\beta}$.

As in |4|, we apply Hadamard's Three Circles Theorem on the function $F(\xi+\lambda w)$ (λ-complex variable) for the circles with radiuses $1, \rho, \bar{R}$ and

$$\gamma := \frac{\log(\bar{R}/\rho)}{\log \bar{R}} = \frac{1}{\beta+1}.$$

All the time, w is a complex parameter. So we have

$$\sup\{|F(\xi+w)|;\ |w| \leq 1\} \geq \tag{14}$$
$$\geq (\sup\{|F(\xi+\rho w)|;\ |w| \leq 1\})^{1+\beta} / (\sup\{|F(\xi+Rw)|;|w| \leq 1\})^{\beta}.$$

Using (9) we obtain

$$|F(\xi+\bar{R}w)| = |F(\xi+\bar{R}\cdot \mathrm{Re}w + i\cdot\bar{R}\cdot \mathrm{Im}w)| \leq$$
$$\leq c\cdot(1+|\xi|)^n\cdot(1+\bar{R})^n\cdot\exp(M_p^*(\bar{R})) \leq c'\cdot c\cdot(1+|\xi|)^n\cdot\exp(2\cdot M_p^*(\bar{R}))$$

where we have put $c' := \sup\{(1+\bar{R})^n \exp(-M_p^*(\bar{R}));\ \bar{R} \in R\} < \infty$.

Since we have construced $\bar{R}$ so that $M_p^*(\bar{R}) = M_q^*(\rho/A_1)$ we have

$$\sup\{|F(\xi+Rw)|;\ |w| \leq 1\} \leq C\cdot(1+|\xi|)^{n+2} \tag{15}$$

for some $C > 0$. Returning to (14) using (11) we obtain the statement for $|\xi| \geq L$, because β is bounded.

Using the Maximum Principle we obtain for $|\xi| \leq L$

$$\sup\{|F(\xi+w)|;\ |w| \leq 1\} \geq C_1 > 0$$

and this togerher with (15) gives that $F(\xi)$ is extremely slowly decreasing.

Let us remark that after the condition (B) Theorem 5 is superfluous.

At last in Theorem 7 there is a miss print, namely $S \in O_C'(H'\{M_p\})$.

We are indebted to Olaf von Grudzinski who noticed that β must not be bounded without additional conditions.

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REZIME

ISPRAVKA RADA "REŠIVOST KONVOLUCIJE JEDNAČINE U $H'\{M_p\}$"

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ON A CLASS OF SPACES OF THE TYPE $S'\{M_p(x,q)\}$

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ABSTRACT

We analyze the structure of the space $\sigma\{M_p\}$ and $\sigma'\{M_p\}$. Under certain conditions on the matrix $\{C_{p,q}\cdot\exp(m_p(x))\}$ we investigate relations between the space $\sigma'\{M_p\}$ and some spaces of ultradistributions. Also we investigate the Fourier transformation on the spaces $\sigma\{M_p\}$ and $\sigma'\{M_p\}$.

1. INTRODUCTION

The spaces of the type $S'\{M_p(x, q)\}$ were introduced in [10], though some examples of such spaces were analyzed already in [1]. In [9] a class of spaces of the type $S'\{M_p(x, q)\}$ was investigated.

In this paper we shall observe a class of spaces of the type $S'\{M_p(x, q)\}$ denoted by $\sigma'\{M_p(x, q)\}$ (short. $\sigma'\{M_p\}$ or σ') for $M_p(x, q) = C_{p,q}\exp(m_p(x))$, $(p,q)\in \mathbb{N}\times\mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Throughout the paper $\{C_{p,q};\ (p,q)\in\mathbb{N}\times\mathbb{N}_0\}$ (short. $\{C_{p,q}\}$) denotes an infinite matrix of positive numbers and $\{m_p(x),\ p\in N\}$ (short. $\{m_p(x)\}$) denotes a sequence of functions. The properties of $\{C_{p,q}\}$ and $\{m_p(x)\}$ will be given later.

We shall analyze the structure of the spaces σ and σ'. The elements of σ' we shall call "exponential ultradistributions". Particularly, we shall prove that under certain conditions

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the space of test functions $\sigma\{M_p\}$ is sufficiently rich and that $\sigma'\{M_p\}$ is a subspace of the space of ultradistributions $\mathcal{D}'^{(N_q)}$ ([3]) for a corresponding sequence $\{N_q\ ;\ q \in \mathbb{N}_o\}$. We shall obtain a representation theorem for exponential ultradistributions. As well, we shall define the space of entire analytic functions on the complex plane which is the Fourier transformation of the space $\sigma\{M_p\}$. This will enable us to define the Fourier transformation of exponential ultradistributions.

2. SPACES $\sigma\{M_p\}$ AND $\sigma'\{M_p\}$

Let $\{C_{p,q}\}$ be an infinite matrix with positive numbers. For this matrix we suppose:

(C.1) $C_{p,q} \leq C_{p+1,q}$ for every $(p,q) \in \mathbb{N} \times \mathbb{N}_o$;

(C.2) For every $p \in \mathbb{N}$ the sequence $\{C_{p,q},\ q \in \mathbb{N}_o\}$ monotonically tends to zero when $q \to \infty$.

(C.3) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$, $p' > p$, such that for every $\varepsilon > 0$ there exists $q_o(\varepsilon) \in \mathbb{N}_o$ with the property $C_{p,q} \leq \varepsilon C_{p',q}$ for $q \geq q_o(\varepsilon)$.

(C.3) makes that (C.1) is superfluous in the theory of spaces σ an σ'. We assume that (C.1) holds only to make the whole theory easier:

In order to have the differentiation as an inner operation in σ' we shall suppose as well:

(C.4) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$, such that $\sup\{C_{p,q}/C_{p',q+1}\ ;\ q \in N_o\} < \infty$.

Let $\{\mu_p(t);\ p \in \mathbb{N}\}$, $t \geq 0$, be a sequence of continuous increasing functions which satisfy: $\mu_p(0) = 0$, $\mu_p(\infty) = \infty$ and $\mu_p(t) \leq \mu_{p+1}(t)$ for every $t \geq 0$, $p \in \mathbb{N}$. Putting

$$m_p(t) = \int_0^{|t|} \mu_p(u)du,\ p \in \mathbb{N},\ t \in \mathbb{R},$$

we obtain another sequence of functions. Every $m_p(t)$, $p \in \mathbb{N}$, is an even convex function which increases to infinity faster than any linear function when $|t| \to \infty$. This implies that its dual function in the sense of Young

$$\tilde{m}_p(y) := \int_0^{|y|} \mu_p^{-1}(t)dt = \sup\{|t\cdot y| - m_p(t);\ t \in \mathbb{R}\}$$

is finite for arbitrary $y \in \mathbb{R}$; $\mu_p^{-1}(t)$, $t \geq 0$ is the inverse function of $\mu_p(t)$ (see [2]). We suppose also in the sequel that the following condition (introduced in [6]), holds :

(A) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$ such that

$$m_p(pt) \leq m_{p'}(t) \text{ holds for } |t| \geq p'.$$

We denote by $\{m_{p,1}(x_1)\},\ldots,\{m_{p,s}(x_s)\}$, $p \in \mathbb{N}$, the sequences of functions obtained from the corresponding sequences $\{\mu_{p,1}(x_1)\},\ldots,\{\mu_{p,s}(x_s)\}$ in the above construction, and we put

$$m_p(x) = m_{p,1}(x_1) + \ldots + m_{p,s}(x_s) \ ,\quad x = (x_1,\ldots,x_s) \in \mathbb{R}^s,$$

Since the sequences $\{m_{p,i}(x_i)\}$, $i = 1,\ldots,s$ satisfy (A), this condition (in an obvious interpretation) holds also for $\{m_p(x)\}$.

Further on in the paper we shall put

$$M_p(x, q) = C_{p,q} \exp(m_p(x)),\ p \in \mathbb{N},\ q \in \mathbb{N}_o,\ x \in \mathbb{R}^s .$$

DEFINITION 1. *The vector space of smooth functions on* $\mathbb{R}$ *such that for every* $p \in \mathbb{N}$

$$\gamma_p(\phi) := \sup\{|\phi^{(q)}(x)| M_p(x,|q|);\ x \in \mathbb{R}^s,\ q \in \mathbb{N}_o^s\} < \infty$$

is denoted by $\sigma\{M_p(x, q)\}$ *(short.* $\sigma\{M_p\}$*) . The topology in the space* $\sigma\{M_p\}$ *is given by the sequence of norms* $\{\gamma_p;\ p \in \mathbb{N}\}$. *(As usual,* $|q| = q_1 + \ldots + q_s$ *where* $q = (q_1,\ldots,q_s)$.*).*

In the usual manner (see [1]) one checks that a sequence $\{\phi_n(x)\}$ from $\sigma\{M_p\}$ converges to $\phi \in \sigma\{M_p\}$ iff on every compact set $K \subset \mathbb{R}$ and every $q \in \mathbb{N}_o^s$ the sequence $\{\phi_n^{(q)};\ n \in \mathbb{N}\}$ converges uniformly to $\phi^{(q)}$ and for every $p \in \mathbb{N}$ there exists $C_p > 0$ such that $\gamma_p(\phi_n) \leq C_p$, for every $n \in \mathbb{N}$.

PROPOSITION 1. *Let* $\phi \in \sigma\{M_p\}$. *Then*

(i) $\lim\limits_{|q|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ x\in \mathbb{R}^s\} = 0$

(ii) $\lim\limits_{|x|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ q\in \mathbb{N}_o^s\} = 0$

P r o o f. (i) follows from (C.3) and (ii) follows from the fact that $m_p(x) \to\infty$ if $|x|\to\infty$.

We denote by σ_p, $p\in N$, a subspace of $C^\infty(\mathbb{R}^s)$ such that $\phi\in\sigma_p$ iff

$$\gamma_p(\phi) < \infty,\ \lim_{|q|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ x\in\mathbb{R}^s\} = 0 \text{ and}$$

$$\lim_{|x|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ q\in\mathbb{N}_o^s\} = 0.$$

THEOREM 1. (i) *The space* σ_p *is a Banach space.*
(ii) *The space* $\sigma\{M_p\}$ *is a Frechet-Schwartz space.*

P r o o f. (i) Let $\gamma_p(\phi_\nu - \phi_\mu) < \varepsilon$ if $\nu,\mu \geq N(\varepsilon)$ and let $\phi\in C^\infty(\mathbb{R}^s)$ be the limit of the sequence $\{\phi_\nu\}$.

We prove that $\phi\in\sigma_p$. Clearly $\gamma_p(\phi) < \infty$ holds. We want to prove the remaining properties of ϕ. First we prove that for every $q\in\mathbb{N}_o^s$

(a) $\sup\{M_p(x,|q|)|\phi_\nu^{(q)} - \phi^{(q)}(x)|;\ x\in\mathbb{R}^s\} \to 0$ as $\nu\to\infty$.

If $\nu,\mu_o > N(\varepsilon)$ we have

(b) $\sup\{M_p(x,|q|)|\phi_\nu^{(q)}(x)|;\ x\in\mathbb{R}^s\setminus K\} \leq$

$\leq \sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x\in\mathbb{R}^s\setminus K\} + \varepsilon$,

where $K = B(0,\rho)$ is the closed ball with radius $\rho > 0$. If $\nu\to\infty$ we obtain

$$\sup\{M_p(x,|q|)|\phi^{(q)}(x)|;\ x\in\mathbb{R}^s\setminus K\} \leq$$

$$\leq \sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x\in\mathbb{R}^s\setminus K\} + \varepsilon .$$

For every compact set $K \subset \mathbb{R}^s$ there exists $N(\varepsilon, K)$ such that

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in K\} < \varepsilon \text{ if } \nu > N(\varepsilon,K).$$

because $\phi_\nu^{(q)}$ converges uniformly to $\phi^{(q)}$ on K.

Let $\mu_o > N(\varepsilon)$ and let ρ be chosen such that for $K = B(0,\rho)$ the following estimate holds

$$\sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} < \varepsilon \quad .$$

Therefore taking $\nu \geq \nu_o(q) = \max\{N(\varepsilon),\ N(\varepsilon, B(0,\rho))\}$ we have

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq$$

$$\leq \sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in K\} +$$

$$+ \sup\{M_p(x,|q|)\,(|\phi_\nu^{(q)}(x)| + |\phi^{(q)}(x)|);\ x \in \mathbb{R}^s \setminus K\} \leq$$

$$\leq \varepsilon + 2\varepsilon + 2\varepsilon = 5\varepsilon \ .$$

Thus we proved (a).

Since $\phi_{\mu_o} \in \sigma_p$, there exists $N_o(\varepsilon)$ such that

$$\sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s\} < \varepsilon \quad \text{if} \quad |q| > N_o(\varepsilon) \ .$$

From (b) we obtain that $\nu > N(\varepsilon)$ and $|q| > N_o(\varepsilon)$ imply

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x)|;\ x \in \mathbb{R}^s\} < 2\varepsilon \ .$$

For a fixed $q \in N_o^s$ and $\nu(q) > \nu_o(q)$ we have

$$\sup\{M_p(x,|q|)\,|\phi^{(q)}(x) - \phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\} < \varepsilon \ .$$

Thus, from

$$\sup\{M_p(x,|q|)\,|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq$$

$$\leq \sup\{M_p(x,|q|)\,|\phi^{(q)}(x) - \phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\} +$$

$$+ \sup\{M_p(x,|q|)|\phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\}$$

we obtain that

$$\lim_{|q|\to\infty} \sup\{M_p(x,|q|)\,|\phi^{(q)}(x)|;\ x\in\mathbb{R}^s\} = 0\ .$$

The proof of

$$\lim_{|x|\to\infty}\{\sup M_p(x,|q|)\,|\phi^{(q)}(x)|;\ q\in\mathbb{N}_o^s\} = 0$$

may be derived in a similar way by observing separately this supremum for $|q| \le q_o$ and $|q| > q_o$ for a suitable $q_o \in \mathbb{N}_o$.

(ii) Proposition 1 implies that $\sigma\{M_p\} = \bigcap_{p=1}^{\infty} \sigma_p$. Let p" be an integer such that $p'' \ge p'$ where p' is an integer which corresponds to given $p\in\mathbb{N}$ in condition (C.3). From condition (A) it follows that we may choose p" such that $\exp(m_p(x) - m_{p''}(x)) \to 0$ as $|x|\to\infty$. We shall show that the inclusion mapping $\sigma_{p''} \to \sigma_p$ is compact. For the proof we shall use an idea from [1].

Let $\{\phi_\nu\}$ be a bounded sequence in $\sigma_{p''}$. We denumerate the set $\mathbb{N}_o^s$ by putting $e_1=(1,\ ,\dots,0)\to 1$, $e_2=(0,1,\dots,0)\to 2,\dots$ etc. By $\{K_n\}$ we denote a sequence of compact subsets of $\mathbb{R}^s$ such that

$$K_n \subset \overset{o}{K}_{n+1},\ n\in\mathbb{N},\quad \bigcup_{n=o}^{\infty} K_n = \mathbb{R}^s \quad \text{and}$$

$$\sup\{\exp(m_p(x) - m_{p''}(x));\ x\in\mathbb{R}^s\setminus K_n\} < \varepsilon_n$$

where the sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, monotonically tends to zero ($\overset{o}{K}_n$ is the interior of K_n).

The functions $|\phi_\nu^{(e_1)}(x)|$, $\nu\in\mathbb{N}$, are uniformly bounded on K_1. Hence, by virtue of the Arzela theorem, there exists a subsequence $\{\phi_{\nu,1}\}$ of $\{\phi_\nu\}$ which converges uniformly on K_1. Because of the uniform boundedness of the functions $|\phi_{\nu,1}^{(e_1)}(x)|$, $\nu,1 = \nu_1\in\mathbb{N}$, on K_2, according to the same Arzela theorem there exists a sub-sequence $\{\phi_{\nu,2}\}$ of $\{\phi_{\nu,1}\}$ such that $\{\phi_{\nu,2}^{(e_2)}\}$ converges uniformly on K_2. Continuing in this manner, and then applying a diagonalization process we obtain a sequence $\{\phi_{\nu\nu}\}$.

As $\{\phi_{\nu,\nu}^{(q)}\}$ converges to $\phi^{(q)}$ on every compact set $K \subset \mathbb{R}^s$ and $\gamma_{p'}(\phi_{\nu,\nu}) \leq M$ we have that $\gamma_{p''}(\phi) \leq M$ because of

$$|\phi_{\nu,\nu}^{(q)}(x)| \leq (M / C_{p'',|q|}) \exp(-m_{p''}(x))$$

which holds on every compact set $K \subset \mathbb{R}^s$.

We shall show that $\phi \in \sigma_p$ and that $\{\phi_{\nu,\nu}\}$ converges to ϕ in σ_p.

For a fixed $q \in \mathbb{N}_o^s$ we have

(c) $$\sup\{C_{p,|q|} \exp(m_p(x)) |\phi^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K_n\} \leq$$

$$\leq \varepsilon_n (C_{p,|q|} / C_{p'',|q|}) \sup\{C_{p'',|q|} \exp(m_{p''}(x)) |\phi^{(q)}(x)|;$$

$$x \in \mathbb{R}^s \setminus K_n\} \leq \varepsilon_n .$$

where ε_n is from (C.3).

Let $\{\varepsilon_n'\}$ be a sequence of real numbers which tends to zero and let $q_o(\varepsilon_n')$, $n \in \mathbb{N}$, be the corresponding numbers from condition (C.3). From the inequality

$$\sup\{C_{p,|q|} \exp(m_p(x)) |\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq$$

$$\leq \varepsilon_n' \cdot \exp(m_p(x) - m_{p''}(x)) \sup\{C_{p'',|q|} \exp(m_{p''}(x)) |\phi^{(q)}(x)|;$$

$$x \in \mathbb{R}^s\} \leq \varepsilon_n' M$$

which holds for $|q| > q_o(\varepsilon_n')$, we obtain that

$$\lim_{|q|\to\infty} \sup\{C_{p,|q|} \exp(m_p(x)) |\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} = 0.$$

This fact, together with (c), implies

$$\lim_{|x|\to\infty} \sup\{C_{p,|q|} |\phi^{(q)}(x)| \exp(m_p(x));\ q \in \mathbb{N}_o^s\} = 0.$$

So we proved that $\phi \in \sigma_p$.

We have

$$\gamma_p(\phi_{\nu,\nu}-\phi) \le \sup\{M_p(x,|q|)|\phi^{(q)}_{\nu,\nu}(x)-\phi^{(q)}(x)|;\ x\in K_n,\ q\in \mathbb{N}^s_o\}+$$
$$+\sup\{M_p(x,|q|)|\phi^{(q)}_{\nu,\nu}(x)-\phi^{(q)}(x)|;\ x\in\mathbb{R}^s\setminus K_n,\ q\in\mathbb{N}^s_o\}\le$$
$$\le \sup\{M_p(x,|q|)|\phi^{(q)}_{\nu,\nu}(x)-\phi^{(q)}(x)|;\ x\in K_n,\ |q|\le q_o(\varepsilon_n')\}+$$
$$+\sup\{M_p(x,|q|)(|\phi^{(q)}_{\nu,\nu}(x)|+|\phi^{(q)}(x)|);\ x\in K_n,\ |q|>q_o(\varepsilon_n')\}+$$
$$+\sup\{M_p(x,|q|)(|\phi^{(q)}_{\nu,\nu}(x)|+|\phi^{(q)}(x)|);\ x\in\mathbb{R}^s\setminus K_n,|q|\le$$
$$\le q_o(\varepsilon_n')\}+\sup\{M_p(x,|q|)(|\phi^{(q)}_{\nu,\nu}(x)|+|\phi^{(q)}(x)|);$$
$$x\in\mathbb{R}^s\setminus K_n,\ |q|>q_o(\varepsilon_n')\}\le\sup\{M_p(x,|q|)|\phi^{(q)}_{\nu,\nu}(x)-$$
$$-\phi^{(q)}|;\ x\in K_n,\ |q|\le q_o(\varepsilon_n')\}+\varepsilon_n' 2M+\varepsilon_n 2M+\varepsilon_n'\varepsilon_n 2M\,.$$

Therefore from the construction of the sequence $\{\phi_{\nu,\nu}(x)\}$ it follows that $\{\phi_{\nu,\nu}\}$ converges to ϕ in σ_p.

We shall turn now to an important example of the space $\sigma\{M_p\}$.

3. IMBEDDING OF $\sigma'\{M_p\}$ INTO ULTRADISTRIBUTIONS

Let

(1) $$c_{p,q}=\frac{p^q}{N_q}$$

where $\{N_q;\ q\in N_o\}$ is an increasing sequence of positive numbers such that the following conditions holds:

(M.1) $N_q^2\le N_{q-1}\,N_{q+1}$, $q\in\mathbb{N}_o$;

(M.2)' There are $A>0$ and $H>0$ such that

$$N_q\le AH^qN_{q+1}\ ,\quad q\in N_o\ ;$$

(M.3)' $\sum\limits_{q=o}^{\infty} N_q/N_{q+1}<\infty$.

(see [3]). Observe that then the matrix $\{\frac{p^q}{N_q}\}$ satisfies the

conditions (C.1)-(C.4). Now putting $M_p(x,|q|) = \frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))$ we come to an example of a space of the type $\sigma\{M_p\}$, i.e. to the space $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$; of course, here $q \in \mathbb{N}_o^s$, $x \in \mathbb{R}^s$.

One checks easily that the space $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ (see [4]) is a subspace of $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$, in fact we shall prove something more:

THEOREM 2. a) *The space* $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ *is a dense subspace of* $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$.

b) *The space* $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ *is sufficiently rich in the sense of* [1].

P r o o f. a) Let $\phi(x)$ be an arbitrary function from $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ (short. $\sigma_1\{M_p\}$). We shall construct a sequence of functions from $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$, which converges to $\phi(x)$ in $\sigma_1\{M_p\}$.

First, let us recall that condition (M.3)′ implies that there exists a nonnegative function with compact support $h(x)$ from $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ such that $h(x) \equiv 1$ on the interval $[-1,1]^s$.

Let $h_n(x) := h(\frac{x}{n})$, $n \in \mathbb{N}$; obviously $h_n(x) \in \mathcal{D}^{(N_q)}(\mathbb{R}^s)$ for every $n \in \mathbb{N}$ and let us put $K_n := \operatorname{supp} h_n$.

We shall use the following inequality

$$\text{(2)} \quad \sup\{\frac{p^{|q|}}{N_{|q|}}|h_n^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq \frac{1}{n^{|q|}}\sup\{\frac{p^{|q|}}{N_{|q|}}|h^{(q)}(x)|;\ x \in \mathbb{R}^s\}$$

for every $p,n \in \mathbb{N}$, $q \in \mathbb{N}_o^s$.

We prove now that the sequence $\{h_n(x)\phi(x)\}$ converges to $\phi(x)$ in the sense of $\sigma_1\{M_p\}$. It is clear that for every compact set $K \subset \mathbb{R}^s$ the sequence $\{(h_n(x)\phi(x))^{(q)};\ n \in \mathbb{N}\}$ converges uniformly to $\phi^{(q)}(x)$ on K. So, we have yet to prove that for every $p \in \mathbb{N}$ there exists $C_p > 0$ with the property

(3) $$\gamma_p(h_n(x)\phi(x)) \le C_p .$$

In fact, from the inequality $N_r N_{q-r} \le N_o N_q$, $0 \le r \le q$, $r,q \in \mathbb{N}_o$, which follows from (M.1), we have

$$\sup\{|(h_n(x)\phi(x))^{(q)}|\exp(m_p(x));\ x \in \mathbb{R}^s\} \le$$

$$\le \sum_{r\le q} \binom{q}{r} \sup\{|h_n^{(r)}(x)|;\ x \in K_n\} \sup\{|\phi^{(q-r)}(x)| \cdot$$

$$\cdot \exp(m_{2p}(x));\ x \in K_n\} \le \sum_{r\le q} \binom{q}{r} \frac{N_{|r|}N_{|q-r|}}{(2p)^{|q-r|}} \sup\{|h_n^{(r)}(x)| \cdot$$

$$\cdot \frac{2p^{|r|}}{N_{|r|}};\ x \in K_n\}\gamma_{2p}(\phi) \le \frac{N_o N_{|q|}}{(2p)^{|q|}} \sum_{r\le q} \binom{q}{r} \sup\{|h^{(q)}(x)| \cdot$$

$$\cdot \frac{(2p)^q}{N_q};\ x \in \mathbb{R},\ q \in \mathbb{N}_o\}\gamma_{2p}(\phi) \le C_p \frac{N_{|q|}}{p^{|q|}}$$

for some $C_p \ge 0$ (as usual, $\binom{q}{r} = \binom{q_1}{r_1}\dots\binom{q_s}{r_s}$, $r \le q \iff r_i \le q_i$, $i=1,2,\dots,s$).

b) We shall check all three conditions from the Lemma on page 236 in [1]. We already know that there exists a nontrivial function in $\sigma_1\{M_p\}$. The translation-invariance of the space $\sigma_1\{M_p\}$ follows from condition (A). In fact, by (A) for given $t \in \mathbb{R}^s$ and $p \in \mathbb{N}$ there exists a $p' \in \mathbb{N}$ such that $m_p(x) \le m_{p'}(x-t)$ for $|x|$ sufficiently large. For $\phi \in \sigma_1\{M_p\}$, this implies

$$\gamma_p(\phi(x-t)) = \sup\{|\phi^{(q)}(x-t)|\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x));\ x\in\mathbb{R}^s, q\in\mathbb{N}_o^s\}$$

$$\leq C\cdot\sup\{|\phi^{(q)}(x)|\frac{p^{|q|}}{N_{|q|}}\exp(m_{p'}(x));\ x\in\mathbb{R}^s,\ q\in\mathbb{N}_o^s\} < \infty$$

for some $C > 0$ which does not depend on x and q. At last, we must show that for arbitrary $t\in\mathbb{R}^s$, the function $\phi(x)\exp i(x,t)$ is in $\sigma_1\{M_p\}$, provided that $\phi\in\sigma_1\{M_p\}$. (As usual, $(x,t) \doteq x_1t_1 + \ldots + x_st_s$). We have

$$\sup\{|D^q(\phi(x)\exp i(x,t))|\exp(m_p(x));\ x\in\mathbb{R}^s\} \leq$$

$$\leq \sum_{r\leq q}\binom{q}{r}\sup\{|\phi^{(q-r)}(x)|\frac{(2p)^{|q-r|}}{N_{|q-r|}}\exp(m_{2p}(x));\ x\in\mathbb{R}^s\}$$

$$\cdot\frac{(2p|t|)^{|q|}}{(2p)^{|q|}N_{|r|}}N_{|r|}N_{|q-r|} \leq \gamma_{2p}(\phi)\frac{N_oN_q}{(2p)^q}\sum_{r\leq q}\binom{q}{r}\frac{(2p|t|)}{N_{|r|}} \leq$$

$$\leq C_1\sup\{\frac{|2pt|^{|q|}}{N_{|q|}};\ q\in\mathbb{N}_o^s\}\cdot\frac{N_{|q|}}{p^{|q|}} \leq C_2\cdot\frac{N_{|q|}}{p^{|q|}} < \infty$$

since $\sup\{\frac{|2pt|^{|q|}}{N_{|q|}};\ q\in N_o^s\} < \infty$ in view of (M.3)$'$; C_1 and C_2 are positive constants which do not depend on x or q.

This theorem shows that $\sigma'\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ (the dual of the space $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p))\}$) is a subspace of a space of ultradistributions $\mathcal{D}^{(N_q)}{}'(\mathbb{R}^s)$.

A sufficient condition which implies that the space $\sigma'\{M_p\}$ is a subspace of a space of ultradistributions is given in the following theorem. Its proof is similar to that of Theorem 2, so we omit it.

THEOREM 3. *Let us suppose for the matrix* $\{C_{p,q}\}$ *that the following condition holds as well:*

(4) *For every* $p \in \mathbb{N}$ *there exist* $p' \in \mathbb{N}$, $p' > p$, *and* $K_{p,p'} > 0$ *such that*

$$2^q C_{p,q} \leq K_{p,p'} \cdot C_{p',q-r} \cdot \frac{(p')^r}{N_r}, \quad r,q \in \mathbb{N}_0, \quad r \leq q.$$

If $\{N_q\}$ *satisfies* (M.1) *and* (M.3)′, *then.*

a) *The space* $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ *is a dense subspace of the space* $\sigma\{M_p\}$;

b) *The space* $\sigma\{M_p\}$ *is sufficiently rich (in the sense of* [

4. A STRUCTURAL THEOREM FOR $\sigma'\{M_p\}$

It is proved in [9] that a linear functional f on $S\{M_p(x, q)\}$ is continuous iff there exists a $p \in \mathbb{N}$ and a sequence of measures $\{f_q;\ q \in \mathbb{N}_0^s\}$ on $\mathbb{R}^s$ such that

$$(5) \quad \sum_{q \in \mathbb{N}_0^s} (\text{total variation of } f_q) < \infty$$

and for every $\phi \in S\{M_p\}$

$$(6) \quad \langle f,\phi\rangle = \sum_{q \in \mathbb{N}_0^s} (-1)^{|q|} \cdot \int_{\mathbb{R}^s} M_p(x,|q|)\cdot \phi^{(q)}(x)\,df_q .$$

Yamanaka proved this theorem under conditions which are all satisfied in the case $M_p(x,|q|) = C_{p,|q|}\exp(m_p(x))$; i.e., the representation (6), under the convergence of the series (5), is valid for the elements from $\sigma'\{M_p\}$. However, for this space we shall obtain a somewhat more precise structural theorem.

THEOREM 4. *A linear functional* f *on* $\sigma\{M_p\}$ *is continuous iff there exists a* $p \in \mathbb{N}$ *and a sequence of functions from* $L_\infty^{loc}(\mathbb{R}^s) : \{f_q(x);\ q \in \mathbb{N}_0^s\}$ *such that*

$$(5') \quad \sum_{q \in \mathbb{N}_o^s} \text{ess sup}\{|f_q(x)|;\ x \in \mathbb{R}^s\} < \infty$$

and for every $\phi \in \sigma\{M_p\}$

$$(6') \quad \langle f,\phi\rangle = \sum_{q \in \mathbb{N}_o^s} C_{p,q}(-1)^{|q|} \cdot \int_{\mathbb{R}^s} \exp(m_p(x)) \cdot \phi^{(q)}(x) \cdot$$

$$\cdot\ f_q(x)dx\ .$$

P r o o f. First we have to prove that the sequence of norms $\{\gamma_p\}$ is equivalent to the sequence of norms $\{\eta_p\}$ where

$$\eta_p(\phi) := \sup\{\int_{\mathbb{R}^s} M_p(x,|q|)|\phi^{(q)}(x)|dx;\ q \in \mathbb{N}_o^s\},\ p \in \mathbb{N}\ .$$

In [6] we proved that condition (A) implies condition (N) ([1]) for every sequence $\{\exp(m_{p,i}(x_i)\}$, $i=1,2,\dots,s$. This fact is crucial for the proof of equivalence of sequences $\{\gamma_p\}$ and $\{\eta_p\}$. In ([7], this Journal) we discuss more about condition (N). Thus, using arguments of the proof of ([7] Lemma 3. (i)) and remarks given in ([7], part 2) one can prove that the sequences $\{\gamma_p\}$ and $\{\eta_p\}$ are equivalent.

For a fixed $p \in N$ we denote by σ_{1p} the normed space defined in the following way

$$\phi \in \sigma_{1p} \text{ iff } \phi \in C^\infty(\mathbb{R}^s)\mu_p(\phi) < \infty \text{ and}$$

$$\lim_{|q|\to\infty} \int_{R^s} M_p(x,|q|)|\phi^{(q)}(x)|dx = 0\ ,$$

Similarly to the proof of Theorem 1 (i) one can prove that σ_{1p}, $p \in \mathbb{N}$, are (B) spaces, $\sigma_{11} \supset \sigma_{12} \supset \dots$ and that the norms $\{\eta_p\}$ are pairwisely compatible. If we denote by σ^p the completion of the space σ according to the norm η_p, $p \in \mathbb{N}$, from [1] p.35 we obtain

$$\sigma' = \bigcup_{p=1}^{\infty} (\sigma^p)'$$

It means that any element f from σ' may be extended from the space σ onto the space σ^p (for some p); this element from $(\sigma^p)'$ let us denote also by f. The σ^p is a closed subspace of the space σ_{1p}. By Hahn-Banach Theorem f may be continuously extended from σ^p on σ_{1p} to be continuous. Contrary, a restriction of any element from σ'_{1p} on σ^p belongs to $(\sigma^p)'$. Since we want to give a representation theorem for the elements from σ', by the given explanations it is enough to prove a representation theorem for elements from σ'_{1p}.

We denote by Γ the subspace of $\prod_{q\in N_o^s} L^1(\mathbb{R}^s)$ defined in the following way

$$\psi = (\phi_q) \in \Gamma \text{ iff } \|\psi\| := \sup\{ \int_{\mathbb{R}^s} |\phi_q(x)|dx;\ q \in \mathbb{N}_o^s \} < \infty$$

$$\text{and } \lim_{|q|\to\infty} \int_{\mathbb{R}^s} |\phi_q(x)|dx = 0 .$$

The space σ_{1p} is isometrically isomorphic to a subspace of Γ, $\Gamma_p = u(\sigma_{1p})$, where u is the mapping defined in the following way

$$\sigma_{1p} \ni \phi \to u(\phi) = (M_p(x,|q|)\cdot \phi^{(q)}(x)) \in \Gamma_p .$$

If $f \in \sigma'_{1p}$ then by

$$\langle \tilde{f},\psi\rangle := \langle f, u^{-1}(\psi)\rangle , \quad \psi \in \Gamma_p ,$$

an element from Γ'_p is defined. By Hahn-Banach Theorem $\tilde{f}$ may be extended on Γ to be an element from Γ'; let us denote this element by F. It is known (see [9] or [4]) that if $F \in \Gamma'$ then there exist functions f_q, $q \in \mathbb{N}_o^s$, from $L^\infty(\mathbb{R}^s)$ such that

$$\langle F,\psi\rangle = \sum_{q\in \mathbb{N}_o^s} \int_{\mathbb{R}^s} f_q(x)\phi_q(x)dx, \quad \psi = (\phi_q) \in \Gamma \quad \text{and}$$

$$\sum_{q\in \mathbb{N}_o^s} \operatorname{ess\,sup}\{|f_q(x)|;\ x \in \mathbb{R}^s\} < \infty$$

It means that on σ_{1p} we have

$$<f,\phi> = <\tilde{f},u(\phi)> = \sum_{q\in \mathbb{N}_0^s} \int_{\mathbb{R}^s} f_q(x)M_p(x,|q|)\phi^{(q)}(x)dx =$$

$$= < \sum_{q\in \mathbb{N}_0^s} (-1)^{|q|}(f_q(x)M_p(x,|q|))^{(q)}, \phi(x)> .$$

We obtain that $f \in \sigma'_{1p}$ iff f is of the form

(7) $$f = \sum_{q\in \mathbb{N}_0^s} (-1)^{|q|}(f_q(x)M_p(x,|q|))^{(q)}$$

such that

(8) $$\sum_{q\in \mathbb{N}_0^s} \text{ess sup}\{|f_q(x)|;\ x\in \mathbb{R}^s\} < \infty$$

where the series in (7) converges weakly in σ_{1p}.

Let us prove now a more suitable representation theorem.

THEOREM 5. *A linear functional* f *on* $\sigma\{M_p\}$ *is continuous iff there exist* $p_1 \in \mathbb{N}$ *and continuous functions* $F_q(x)$, $q \in \mathbb{N}_0^s$, *on* $\mathbb{R}^s$ *with the property* $\sum_{q\in \mathbb{N}_0^s} \sup\{|F_q(x)|,\ x\in \mathbb{R}^s\} < \infty$ *such that for every* $\phi \in \sigma\{M_p\}$

(9) $$<f,\phi> = \sum_{q\in \mathbb{N}_0^s} C_{p_1,|q|}(-1)^{|q|} \int_{\mathbb{R}^s} \exp(m_{p_1}(x))\phi^{(q)}(x)F_q(x)dx$$

P r o o f. By what was said before, the condition is sufficient. Conversely, from representation (6′) we obtain that there exist a natural number p and bounded measurable functions f_q such that (5′) holds with the property

$$<f,\phi> = \sum_{q\in \mathbb{N}_0^s} C_{p,|q|}(-1)^{|q|} \int_{\mathbb{R}^s} \exp(m_p(x))\phi^{(q)}(x)f_q(x)dx$$

$\phi \in \sigma\{M_p\}$, or symbolically

$$(10) \qquad f = \sum_{q \in \mathbb{N}_0^s} c_{p,|q|} D^q(\exp(m_p(x)) f_q(x)).$$

Let us choose $p_1 \in \mathbb{N}$ such that the function $x \cdot \exp(m_p(x) - m_{p_1}(x))$ is bounded (see (N)) and condition (C.4) holds. Then we obtain

$$(10)' \quad f = \sum_{q \in \mathbb{N}_0^s} c_{p_1,|q+1|} D^{q+1}(\exp(m_{p_1}(x) F_{q+1}(x)))$$

where

$$F_{q+1}(x) := \frac{c_{p,|q|}}{c_{p_1,|q+1|}} \exp(-m_{p_1}(x)) \int_0^x \exp(m_p(t)) f_q(t) dt, \; q \in \mathbb{N}_0^s$$

are bounded continuous functions or $\mathbb{R}^s$. Since

$$\sum_{q \in \mathbb{N}_0^s} \sup\{|F_{q+1}(x)|;\, x \in \mathbb{R}^s\} \le \sup\{\frac{c_{p,q}}{c_{p,|q+1|}};\, q \in \mathbb{N}_0\} \cdot$$

$$\sup\{|x| \exp(m_p(x) - m_{p_1}(x));\, x \in \mathbb{R}^s\} \cdot \sum_{q \in \mathbb{N}_0^s} \operatorname{ess\,sup}\{|f_q(x)|;$$

$$x \in \mathbb{R}^s\} < \infty ,$$

the relation $(10)'$ is the desired representation of f.

5. FOURIER TRANSFORMATION ON $\sigma\{M_p\}$

In this section we define the space of entire analytic functions Ψ such that $F(\phi) = \Psi$ for some $\phi \in \sigma\{M_p\}$; as usual F stands for the Fourier transform. This enables us to define, through the Parseval equality, the Fourier transform of the elements from $\sigma'\{M_p\}$.

We denote by $\zeta = \xi + i\eta$ the s-dimensional complex variable, $\zeta = (\zeta_1, \ldots, \zeta_s)$ where $\zeta_k = \xi_k + i\eta_k \in \mathbb{C}$, $k = 1, \ldots, s$. As

usual, the scalar product $\langle x,\zeta\rangle$ is $\sum_{k=1}^{s} x_k\zeta_k$ for $x=(x_1,\dots,x_s)\in\mathbb{R}^s$ and $\zeta=(\zeta_1,\dots,\zeta_s)\in\mathbb{C}^s$.

The Fourier transform of a function $\phi\in L^1_{loc}(\mathbb{R}^s)$ is defined by

$$(11)\qquad \hat{\phi}(\zeta) = F(\phi(x))(\zeta) = \int_{\mathbb{R}^s} e^{-i\langle x,\zeta\rangle}\cdot\phi(x)dx$$

provided that this integral converges. First we prove a Lemma.

LEMMA 1. *Let* $\phi\in\sigma\{M_p\}$. *Then the integral* (11) *defines an entire analytic function* $\hat{\phi}(\zeta)$ *of* $\zeta=\xi+i\eta\in\mathbb{C}^s$ *such that*

$$(12)\qquad |\zeta^q\cdot\hat{\phi}(\zeta)| \le \frac{A_p}{C_{p,|q|}}\exp(\tilde{m}_p(\eta)),\ p=1,2,\dots,$$

for some $A_p>0$.

Of course, $\tilde{m}_p(\eta)=\tilde{m}_{p,1}(\eta_1)\dots\tilde{m}_{p,s}(\eta_s)$, $\eta=(\eta_1,\dots,\eta_s)$.

P r o o f. Let us take $\eta_o=(\eta_{o,1},\dots,\eta_{o,s})$, $\eta_{o,k}>0$, $k=1,\dots,s$ and estimate the integral

$$(13)\quad |(-i)^{|q|}\int_{\mathbb{R}^s} x^q\cdot e^{-i\langle x,\zeta\rangle}\phi(x)dx| \quad\text{for}\quad |\eta_k|\le\eta_{o,k},$$

$k=1,\dots,s$ and $q=(q_1,\dots,q_s)\in\mathbb{N}_o^s$.

Since it is less or equal than

$$C\cdot\prod_{k=1}^{s}\int_{\mathbb{R}}(1+|x_k|)^{q_k}e^{x_k\eta_k}e^{-m_{p,k}(x_k)}dx_k \quad\text{for some } C=C(p,q)>0$$

and

$$m_{p,k}(x_k)\ge 3\cdot|x_k|\eta_{o,k}-A_{p,k}\ (A_{p,k}>0),\ k=1,\dots,s$$

we obtain that (13) uniformly converges in any "strip" $\{\xi+i\eta\in\mathbb{C}^s; |\eta|\le\eta_o\}$. This implies that we can differentiate under the integral sign arbitrary many times. This means that $\hat{\phi}(\zeta)$ is an entire analytic function on $\mathbb{C}^s$. Let us prove now (12); we observe first that

$$|\zeta^q\hat{\phi}(\zeta)| = |F(\phi^{(q)}(x))(\zeta)| = |\int_{\mathbb{R}^s} e^{-i<x,\zeta>}\phi^{(q)}(x)dx| .$$

For given $p \in \mathbb{N}$ we choose $p' \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^s} \exp(m_p(x) - m_{p'}(x))dx < \infty$$

(condition (N)). Now if $\phi \in \sigma\{M_p\}$ we have

$$|\zeta^q\hat{\phi}(\zeta)| \leq \frac{C_1}{C_{p',|q|}} \int_{\mathbb{R}^s} \exp\left(\sum_{i=1}^{s} |x_i||\eta_i| - m_{p'}(x)\right)dx \leq$$

$$\leq \frac{C_1}{C_{p',|q|}} \sup\{\exp\left(\sum_{i=1}^{s} |x_i||\eta_i| - m_p(x)\right);\ x \in \mathbb{R}^s\}.$$

$$\int_{\mathbb{R}^s} \exp(m_p(x)-m_{p'}(x))dx = \frac{C_2}{C_{p',|q|}} \cdot \exp(\tilde{m}_p(\eta)) \leq \frac{C_2}{C_{p,|q|}} \exp(\tilde{m}_p(\eta))$$

for some positive constants C_1, C_2 which depend on p but not on $q \in \mathbb{N}_o^s$.

If $C_{p,|q|} = \frac{p^{|q|}}{N_{|q|}}$, $(p,|q|) \in \mathbb{N} \times \mathbb{N}_o$ where $\{N_{|q|};\ |q| \in \mathbb{N}_o\}$ satisfies the conditions (M1), (M2)′ and (M3)′, from this lemma and ([3], Lemma 3.3), directly follows that for $\phi \in \sigma\{\frac{p^{|q|}}{N_{|q|}} \exp(m_p(x)\}$ we have the following statement:

For any p there exists $C_p > 0$ such that

$$|\hat{\phi}(\zeta)| \leq C_p \cdot \exp\{\tilde{m}_p(\eta) - M(p|\eta|)\} ,$$

where $M(\rho)$, $\rho > 0$, is the associated function to $\{N_q\}$,

$$(14) \quad M(\rho) := \sup\{\log \frac{\rho^{|q|} N_{|o|}}{N_{|q|}};\ |q| \in \mathbb{N}_o\} \text{ (see [3])}.$$

Let us prove an inequality in the opposite direction.

LEMMA 2. *Let $\Psi(\zeta)$ be an entire analytic function such that*

$$(15) \quad |\zeta^q||\Psi(\zeta)| \leq \frac{B_p}{C_{p,|q|}} \exp(\tilde{m}_p(\eta)) \quad \text{for } \zeta = \xi + i\eta \in \mathbb{R}^s ,$$

every $(p,q) \in \mathbb{N} \times \mathbb{N}_o$ *and some* $B_p > 0$. *Then the function* ϕ *defined by*

$$(16) \quad \phi(x) := \int_{\mathbb{R}^s} e^{i<x,\xi>} \psi(\xi) d\xi, \quad x \in \mathbb{R}^s$$

is a smooth function on $\mathbb{R}^s$ *which belongs to* $\sigma\{M_p\}$.

Let us observe that for $C_{p,|q|} = \frac{p^{|q|}}{N_{|q|}}$ the inequality (15) can be written as

$$(17) \quad |\Psi(\zeta)| \leq B_p \exp(\tilde{m}_p(\eta) - M(p|\zeta|))$$

provided that (M1), (M2)´ and (M3)´ hold for $\{N_{|q|}, |q| \in \mathbb{N}_o\}$. The function M is defined in (14). So in this case we have a more precise statement.

P r o o f of Lemma 2. The behaviour of $\Psi(\zeta)$ for $|\zeta|$ large implies that this integral defines a smooth function on $\mathbb{R}^s$. Let us take $p \in \mathbb{N}$. First, we replace the real hyperplane $\mathbb{R}^s$ in (16) by the hyperplane

$\mathbb{R}^s + i\eta = \{\xi + i\eta; \xi \in \mathbb{R}^s\}$ (we shall fix $\eta \in \mathbb{R}^s$ later);

in fact, from (15) it follows that

$$\phi(x) = \int_{\mathbb{R}^s} e^{i<x,\xi+i\eta>} \Psi(\xi+i\eta) d\xi \quad \text{and}$$

$$\phi^{(q)}(x) = \int_{\mathbb{R}^s} i^{|q|} (\xi+i\eta)^q \Psi(\xi+i\eta) d\xi \quad (q = (q_1, \dots, q_s) \in \mathbb{N}_o^s)$$

Since by definition $\zeta^q = \zeta_1^{q_1} \cdot \dots \cdot \zeta_s^{q_s}$, $\zeta = (\zeta_1, \dots, \zeta_s) \in \mathbb{C}^s$, using the inequality

$$|\zeta_k|^{q_k} \leq \frac{|\zeta_k|^{q_k} + |\zeta_k|^{q_k+2}}{\zeta_k^2 + 1}, \quad k=1,\dots,s,$$

we obtain

$$|\phi^{(q)}(x)| \leq \exp(-\langle x,\eta\rangle) \int_{\mathbb{R}^s} (|\zeta|^{q} + |\zeta|^{q+\bar{2}}) |\Psi(\zeta)| \cdot$$

$$\cdot \frac{d\xi_1 \ldots d\xi_s}{(\xi_1^2+1)\ldots(\xi_s^2+1)}$$

where $q+\bar{2}$ denotes $(q_1+2,\ldots,q_s+2)$. By assumption we get

$$(18) \quad |\phi^{(q)}(x)| \leq \exp(-\langle x,\eta\rangle)\cdot\Big(\frac{B_p}{C_{p'',|q|}} \exp(\tilde{m}_{p''}(\eta)) +$$

$$+ \frac{B_{p''}}{C_{p'',|q+\bar{2}|}} \exp(\tilde{m}_{p''}(\eta))\Big)$$

where $p'' \in \mathbb{N}$, $p'' > p$, is chosen so that

$$(19) \qquad \sup\Big\{\frac{C_{p,|q|}}{C_{p'',|q+\bar{2}|}} \,;\, |q| \in \mathbb{N}_0\Big\} < \infty \quad \text{(see C.4)}.$$

For $x = (x_1,\ldots,x_s) \in \mathbb{R}^s$, $x_1\cdot\ldots\cdot x_s \neq 0$ we choose each component η_k, $k=1,2,\ldots,s$ of $\eta \in \mathbb{R}^s$ such that $x_k\cdot\eta_k > 0$ for every $k=1,2,\ldots,s$. Taking the infinum by η of the right hand side in (18) we obtain

$$|\phi^{(q)}(x)| \leq \bar{B}_{p''} \exp(-m_{p''}(x)\cdot\Big(\frac{1}{C_{p,|q|}} + \frac{1}{C_{p'',|q+\bar{2}|}}\Big)$$

for some $\bar{B}_{p''} > 0$, which depends also on the sign of x_k, $k=1,\ldots$ $\ldots,s$. Howewer, we see at once that a constant $\bar{B}_{p''} > 0$ can be found which depends only on $p \in \mathbb{N}$. Hence

$$C_{p,|q|} \exp(m_p(x))\cdot|\phi^{(q)}(x)| \leq \bar{B}_{p''}\Big(\frac{C_{p,|q|}}{C_{p'',|q|}} + \frac{C_{p,|q|}}{C_{p'',|q+2|}}\Big)$$

and by (19) we get at last

$$\gamma_p(\phi) < \infty .$$

The space of entire analytic functions which satisfy (15) for every $(p,|q|) \in \mathbb{N} \times \mathbb{N}_o$ we denote by $H\{M_p\}$. From Lemmas 1 and 2 we get

THEOREM 9. *The Fourier transformation is a topological isomorphism between* $\sigma\{M_p\}$ *and* $H\{M_p\}$.

The Fourier transform of $T \in \sigma'\{M_p\}$ is an analytic functional $\hat{T}$ on $H\{M_p\}$. We define it in the usual way:

$$<\hat{T},\hat{\phi}> := 2\pi <T(x),\phi(-x)>, \ \phi \in \sigma\{M_p\}.$$

(Obviously $\phi(-x) \in \sigma\{M_p\}$ if $\phi(x) \in \sigma\{M_p\}$).

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REZIME

O KLASI PROSTORA TIPA $S'\{M_p(x, q)\}$

U radu je analizirana struktura prostora $\sigma\{M_p\}$, $\sigma'\{M_p\}$. Pod odredjenim uslovima za matricu $\{C_{p,q}\cdot\exp(M_p(x))\}$ ispitan je odnos prostora $\sigma'\{M_p\}$ i prostora ultradistribucija. Takodje je ispitana Furijerova transformacija na prostorima $\sigma\{M_p\}$, $\sigma'\{M_p\}$.

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NUCLEARITY OF THE SPACE $\sigma\{M_p(x, q)\}$

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ABSTRACT

The aim of this paper is to prove the nuclearity of the space $\sigma\{M_p(x,q)\}$ under suitable conditions on the matrix $\{M_p(x,q)\}$, $p \in N$, $q \in N_o$. This space is investigated in paper [5] (in this journal), so for notations and notions see [5].

1. In the proof of the nuclearity of the space $K\{M_p\}$ in [2] some conditions were supposed. In this part of the paper we are going to discuss some of them. First, we shall repeat some facts from [1].

Let $\{M_p(x)\}$ be a sequence of continuous functions on R such that:

(1) $$0 < \delta \leq M_1(x) \leq M_2(x) \leq \dots, \quad x \in R.$$

The space $K\{M_p\}$ is defined as the space of smooth functions ϕ such that

$$\|\phi\|_p := \sup\{M_p(x)\,|\phi^{(i)}(x)|;\ \ i \leq p,\ x \in R\} < \infty\ ,\ p \in N\ ;$$

the topology in $K\{M_p\}$ is given by the sequence of norms $\{\|\ \|_p\}$.

Moreover, let us suppose that M_p, $p \in N$, monotonically increase as $|x| \to \infty$ (this means if $|x_1| > |x_2|$, $x_1 \cdot x_2 > 0$, then $M_p(x_1) > M_p(x_2)$) and:

(N) For every $p \in N$ there exists $p' \in N$ such that $r_{p,p'}(x) := M_p(x) M_{p'}^{-1}(x) \in L^1(R)$ and $r_{p,p'}(x) \to 0$ monotonically as $|x| \to \infty$

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(I) For every $p \in N$ and every $k \in N_o$, there exist $p' \in N$ and $B_{p,k} > 0$ such that

$$|M_p^{(k)}(x)| \leq B_{p,k} M_{p'}(x), \quad x \in R.$$

Under these conditions, it is proved in [2] that $K\{M_p\}$ is nuclear.

The supposition that $M_p(x)$, $p \in N$, are smooth functions and that (I) holds, can be changed in some sense with more general conditions. Namely, the following theorem holds.

THEOREM 1. *Let $M_p(x)$, $p \in N$, be a continuous functions on R such that $M_p(x)$ monotonically increase when $|x| \to \infty$. If this sequence satisfies condition* (1) *and*

(T) *There exists $\varepsilon > 0$ such that for every $p \in N$ there exist $p' \in N$ and $K_{p,p'} > 0$ such that*

$$M_p(x) \leq K_{p,p'} M_{p'}(x-\varepsilon) \qquad for \qquad x > K_{p,p'},$$

$$M_p(x) \leq K_{p,p'} M_{p'}(x+\varepsilon) \qquad for \qquad x < -K_{p,p'},$$

then the space $K\{M_p\}$ is equal to the space $K\{N_p\}$ for a suitable sequence of smooth functions $\{N_p(x)\}$ for which (1) *and* (I) *hold.*

If the sequence $\{M_p\}$ satisfies condition (N), *the sequence $\{N_p\}$ satisfies this condition as well.*

P r o o f. For the proof we shall use the following construction ([4]).

Let $\omega_1(x) \in C_o^\infty(R)$, $\operatorname{supp} \omega_1 \subset [0,\varepsilon]$, $\omega_1(x) \geq 0$ and $\int_R \omega_1(x)dx = 1$.

We define the sequence of smooth functions on the interval $[\varepsilon,\infty)$ by

$$N_p(x) := M_p(x) * \omega_1(x) = \int_0^\varepsilon M_p(x-t)\omega_1(t)dt, \quad x \in [\varepsilon,\infty),\ p \in N.$$

So we have

$$(2) \qquad M_p(x-\varepsilon) \leq \bar{N}_p(x) \leq M_p(x), \quad x \in [\varepsilon,\infty),\ p \in N,$$

$$(3) \qquad \bar{N}_p(x) \leq \bar{N}_{p+1}(x), \quad x \in [\varepsilon,\infty),\ p \in N.$$

Similarly, let $\omega_2(x) \in C_o^\infty(R)$, $\mathrm{supp}\,\omega_2 \subset [-\varepsilon,0]$, $\omega_2(x) \geq 0$, $\int_R \omega_2(x)\,dx = 1$, and let

$$\bar{\bar{N}}_p(x) := M_p(x) * \omega_2(x) = \int_{-\varepsilon}^{o} M_p(x-t)\omega_2(t)dt, \quad x \in (-\infty,-\varepsilon], \ p \in N.$$

This sequence satisfies the following inequalities .

(2^o) $\quad M_p(x+\varepsilon) \leq \bar{\bar{N}}_p(x) \leq M_p(x), \quad x \in (-\infty,-\varepsilon], \ p \in N,$

(3^o) $\quad \bar{\bar{N}}_p(x) \leq \bar{\bar{N}}_{p+1}(x), \quad x \in (-\infty,-\varepsilon], \ p \in N.$

There exists a sequence of smooth functions $\{N_p(x)\}$ on R such that: $N_p(x)$ is equal to $\bar{N}_p(x)$ on the interval $[\varepsilon,\infty)$; $N_p(x)$ is equal to $\bar{\bar{N}}_p(x)$ on the interval $(-\infty,-\varepsilon]$; $0 < \theta \leq N_p(x) \leq N_{p+1}(x)$ on the interval $(-\varepsilon,\varepsilon)$, $p \in N$.

Thus we construct the sequence of smooth monotonically increasing functions (as $|x| \to \infty$) $\{N_p\}$, which satisfies (1). From (2), (2^o) and (T) it follows that for any $p \in N$ there exists $p' \in N$ such that

(2^{oo}) $\quad M_p(x)/K_{p,p'} \leq M_{p'}(x-\varepsilon) \leq N_{p'}(x) \leq M_{p'}(x) \quad$ for $\quad x > K_{p,p'}$,

(3^{oo}) $\quad M_p(x)/K_{p,p'} \leq M_{p'}(x+\varepsilon) \leq N_{p'}(x) \leq M_{p'}(x) \quad$ for $\quad x < -K_{p,p'}$,

Thus the spaces $K\{M_p\}$ and $K\{N_p\}$ are the same in the topological sense.

Condition (I) for the sequence $\{N_p(x)\}$ follows from

$$|N_p^{(k)}(x)| \leq \int_o^{\varepsilon} M_p(x-t)|\omega_1^{(k)}(t)|dt \leq M_p(x) \int_o^{\varepsilon} |\omega_1^{(k)}(t)|dt \leq$$

$$\leq CM_p(x) \leq C\,M_{p'}(x-\varepsilon) \leq C\,N_{p'}(x), \quad x \geq K_{pp'},$$

since similar inequalities hold for $x < -K_{p,p'}$.

Let condition (N) holds for the sequence $\{M_p(x)\}$. If $p' \in N$ corresponds to $p \in N$ in (N) let p'' and $K_{p',p''}$ correspond to p' in condition (T). For $x > K_{p',p''}$ we have

$$\frac{N_p(x)}{N_{p''}(x)} \leq \frac{M_p(x)}{M_{p''}(x-\varepsilon)} \leq K_{p',p''} \frac{M_p(x)}{M_{p'}(x)} .$$

Since a similar inequality holds for $x<-K_{p',p''}$, it follows that (N) holds for the sequence $\{N_p(x)\}$.

2. In paper [5] we defind the space $\sigma\{M_p(x,q)\}$ by a suitable matrix $\{c_{p,q}\}$ and a suitable sequence of functions $\{\exp(m_p(x))\}$ where we have constructed the sequence $\{m_p(x)\}$ such that this space may be investigated by a Fourier transformation. Since in this paper a Fourier transformation is not needed, we genaralize the conditions for the matrix $\{M_p(x,q)\}$. Namely, we suppose that

$$M_p(x,q) = M_p(x)c_{p,q}, \quad p\in N, \quad q\in N_o,$$

where $\{c_{p,q}\}$ is a matrix which satisfies some of conditions (C.1), (C.2), (C.3), (C.4) from [5] (see remark about (C.1) in [

(C,1) $c_{p,q}\le c_{p+1,q}$ for every $(p,q)\in N\times N_o$;

(C.2) For every $p\in N$ the sequence $\{c_{p,q};\ q\in N_o\}$ monotonically tends to zero when $q\to\infty$;

(C.3) For every $p\in N$ there exists $p'\in N, p'>p$, such that for every $\varepsilon>0$ there exists $q_o(\varepsilon)\in N$ with the property $c_{p,q}\le\varepsilon\, c_{p',q}$ for $q\ge q_o$;

(C.4) For every $p\in N$ there exists $p'\in N$, such that $\sup\{\frac{c_{p,q}}{c_{p',q+1}};\ q\in N_o\}<\infty$;

and (C.5) (see below), and $\{M_p(x)\}$ is a sequence of functions which satisfies the conditions of Theorem 1. It is clear that the sequence $\{\exp(m_p(x))\}$ from [5] satisfies these conditions.

Let us denote by $E_I^{(c_{p,q})}$, where I is a closed finite interval in R, the space of smooth functions ϕ on I such that

$$\|\phi\|_{p,I}:=\sup\{c_{p,q}|\phi^{(q)}(x)|\ ;\ x\in I,\ q\in N_o\}<\infty,\ p\in N,$$

in which the sequence of norms $\{\|\ \|_{p,I}\}$ defines a topology.

If $\{c_{p,q}\}$, $p\in N$, $q\in N_o$, are of the form

$$c_{p,q} = p^q/N_q, \quad p\in N, \quad q\in N_o,$$

where $\{N_q\}$ is a suitable sequence of positive numbers, then this space becomes the space $E_I^{(N_p)}$ from [3].

The nuclearity of this space follows from the appropriate condition on $\{N_q\}$ ((M.2)´, see [3]). So if we give an appropriate condition on $\{C_{p,q}\}$ we shall obtain the nuclearity of $E_I(C_{p,q})$.

LEMMA 1. *Let for the matrix* $\{c_{p,q}\}$, (C.1) *and* (C.5) *hold, where:*

(C.5) *For every* $p \in N$ *there exists* $p' \in N$, such that

$$\sum_{q \in N_o} c_{p,q}(c_{p',q})^{-1} < \infty .$$

Then the space $E_I^{(c_{p,q})}$ *is nuclear.*

The proof of this Lemma is similar to the proof of Proposition 2.4. from [3], so we have omitted it. Let us only remark that (C.3) follows from (C.5).

LEMMA 2. *The sequence of norms* $\{\| \|_{p,I}\}$ *on* $E_I^{(c_{p,q})}$ *is equivalent to the sequence of norms*

$$\| \phi \|_{p',I} = \sum_{q=o}^{\infty} c_{p,q} \int_I |\phi^{(q)}(x)|^2 dx)^{1/2}, \ p \in N,$$

if (C.1), (C.4) and (C.5) hold.

P r o o f. If $\phi \in E_I^{(c_{p,q})}$, (C.5) implies

$$\|\phi\|'_{p,I} \leq C \ \|\phi\|_{p',I} \quad \text{where } C = \int_I dx \sum_{q \in N_o} c_{p,q}(c_{p',q})^{-1}$$

From the Sobolev Lemma (see [6], Theorem 4.1) it follows:

$$\sup\{|\phi(x)|;\ x \in I\} \leq \sup\{(\int_I |\phi^{(i)}(x)|^2 dx)^{1/2};\ i = 0,1,2\}$$

So we have

$$\sup\{c_{p,q}|\phi^{(q)}(x)|;\ x \in I,\ q \in N_o\} \leq \sup\{c_{p'',q}(\int_I |\phi^{(q)}(x)|^2 dx)^{1/2}$$

$$+ c_{p'',q+1}(\int_I |\phi^{(q+1)}(x)|^2 dx)^{1/2} \frac{c_{p,q}}{c_{p'',q+1}} +$$

$$+ c_{p'',q+2}(\int_I |\phi^{(q+2)}(x)|^2 dx)^{1/2} \frac{c_{p,q}}{c_{p'',q+2}} \quad ; \quad q \in N_o \}$$

where p" corresponds to p′ in (C.4) and p′ corresponds to p also in (C.4). From this inequality it follows that

$$\|\phi\|_{p,I} \le (1+2A)\ \|\phi\|'_{p,I}\ , \quad \text{where}$$

$$A = \sup\left\{ \frac{c_{p,q}}{c_{p'',q+1}} + \frac{c_{p,q}}{c_{p'',q+2}}\ ;\ q \in N_o \right\} .$$

The constant A exists because (C.4) holds.

Let $\{M_p(x)\}$ satisfies the conditions of Theorem 1 and condition (N); $\{c_{p,q}\}$ satisfies the conditions (C.1), (C.2), (C.3) and (C.4). Let us denote by $\{\gamma_{p,M}\}$ the sequence of norms on $\sigma\{M_p(x,q)\}$ defined by $\gamma_{p,M} = \gamma_p$, $p \in N$, where $\{\gamma_p\}$ is the sequence of norms in $\sigma\{M_p(x)c_{p,q}\}$ defined by

$$\gamma_p(\phi) := \sup\{c_{p,q} M_p(x)\,|\phi^{(q)}(x)|;\ q \in N_o,\ x \in R\} .$$

LEMMA 3. *The sequence of norms* $\{\gamma_{p,M}\}$ *is equivalent with the following two sequences of norms*

(i) $\gamma'_{p,M} := \sup\{c_{p,q}\left(\int_R |M_p(x)\phi^{(q)}(x)|^2 dx\right)^{1/2};\ q \in N_o\}, p \in N$

(ii) $\gamma''_{p,N} := \sum_{n=-\infty}^{\infty} \mu_{n,p} \int_n^{n+1} |\phi^{(q)}(x)|^2 dx)^{1/2},\quad p \in N ;$

where

$$\mu_{n,p} = N_p(n+1),\quad n \in Z = -N \cup N_o .$$

P r o o f. (i) It is clear that every norm from the sequence $\{\gamma'_{p,M}\}$ may be majorized by some norm from the sequence $\{\gamma_{p,M}\}$ (see [2] p. 82).

From (N) it follows that for any $q \in N_o$, $N_p(x)|\phi^{(q)}(x)|$ as $|x| \to \infty$.

Using this fact and (I), for $x \in R$ we have

$$|N_p(x)\phi^{(q)}(x)| \le \left|\int_{-\infty}^{x} (N_p(t)\phi^{(q)}(t))'dt\right| \le$$

$$\le \int_{-\infty}^{\infty} |(N_p(x)\phi^{(q)}(x))'|dx \le \int_{-\infty}^{\infty} |N'_p(x)\ \phi^{(q)}(x)|dx + \int_{-\infty}^{\infty} |N_p(x)\ \phi^{(q+1)}(x)|dx$$

$$\le\ B_{p,1} \int_{-\infty}^{\infty} N_p(x)\,|\phi^{(q)}(x)|dx + \int_{-\infty}^{\infty} N_p(x)\,|\phi^{(q+1)}(x)|dx .$$

Multiplying this inequality by $c_{p,q}$,from (C.4) we obtain

(4) $\qquad \gamma_{p,N}(\phi) \leq B_{p,1}\gamma'_{p',N}(\phi) + A\gamma'_{p',N}(\phi)$

where p' corresponds to p in (C.4) and $A = \sup\{c_{p,q}/c_{p',q+1}$, $q \in N_o\}$.

From Theorem 1, more precisely from (2^{oo}) and (3^{oo}) , it follows that the sequence $\{\gamma_{p,M}\}$ on $\sigma\{M_p(x)c_{p,q}\} \equiv \sigma\{N_p(x)c_{p,q}\}$ is equivalent to the sequence $\{\gamma_{p,N}\}$. Using this equivalence for the left hand side of (4), and (2^{oo}) and (3^{oo}) for the right hand side of (4) we obtain that every norm from $\{\gamma_{p,M}\}$ may be majorized by some norm from $\{\gamma'_{p,M}\}$.

(ii) Since the sequences $\{\gamma_{p,M}\}$, $\{\gamma_{p,N}\}$ and $\{\gamma'_{p,N}\}$ are mutually equivalent we ought to prove that $\{\gamma'_{p,N}\}$ and $\{\gamma''_{p,N}\}$ are equivalent. This follows from: condition (I) for $\{N_p(x)\}$; the monotonicity of $N_p(x)$ if $|x| \to \infty$, $p \in N$; condition (T) and

$$\int_R N_p^2(x)|\phi^{(q)}(x)|^2dx = \sum_{n\in Z} \int_n^{n+1} N_p^2(x)|\phi^{(q)}(x)|^2dx .$$

Let us prove this assertion.

If $n \in Z$, $\phi \in \sigma\{N_p(x,q)c_{p,q}\}$ we have

$$\int_n^{n+1} N_p^2(x)|\phi^{(q)}(x)|^2dx \leq \mu^2_{n,p} \int_n^{n+1} |\phi^{(q)}(x)|^2dx ,$$

so $\gamma'_{p,N}(\phi) \leq \gamma''_{p,N}(\phi)$, $p \in N$. $N_p(x)$, $p \in N$ monotonically increase when $|x| \to \infty$, thus for a large enough $|n|$ we have

$$\int_n^{n+1} N^2_{p_1}(x)|\phi^{(q)}(x)|^2dx \geq \mu^2_{n-1,p_1} \int_n^{n+1} |\phi^{(q)}(x)|^2dx \geq$$

$$\geq D_{p,p_1}\mu^2_{n,p} \int_n^{n+1} |\phi^{(q)}(x)|^2dx .$$

where we choose p_1 and $D_{p,p_1} > 0$ such that

(5) $\quad N_{p_1}(x-1) \geq D_{p,p_1}N_p(x) \quad$ for $\quad x > D_{p,p_1}$;

$\quad N_{p_1}(x+1) \geq D_{p,p_1}N_p(x) \quad$ for $\quad x < -D_{p,p_1}$.

The existence of p_1 and D_{p,p_1} follows from (T), if this condition is taken m-times where $m\varepsilon \geq 1$. If the numbers p_1 diverge to infinity the corresponding p in (5) also diverge to infinity. So for a suitable C and any $\phi \in \sigma\{M_p(x,q)\}$ we obtain

$$\gamma''_{p_1,N}(\phi) \geq C\, \gamma'_{p,N}(\phi) .$$

Thus the sequences $\{\gamma'_{p,N}\}$ and $\{\gamma''_{p,N}\}$ are equivalent.

3. Now we are ready to prove the following theorem.

THEOREM 2. *Let* $\{M_p(x)\}$ *be the sequence of functions from Theorem 1 for which* (N) *holds, and let* $\{c_{p,q}\}$ *be a matrix of positive numbers for which* (C.1) — (C.5) *hold. The corresponding space* $\sigma\{M_p(x)c_{p,q}\}$ *is nuclear.*

P r o o f. For the proof we need to check that the conditions from the construction of a nuclear space by a known nuclear space, given in [2] p.p. 80, 81, are satisfied.

If we put $m_{n,p} = \mu_{n,p}$, $n \in Z$, $p \in N$, we have

$$m_{n+1,p} \geq m_{n,p} > 0,\ n \in N_o,\ p \in N,$$

$$m_{n-1,p} \geq m_{n,p} > 0,\ n \in -N,\ p \in N,$$

because $N_p(x)$ increases if $|x|$ increases;

$$m_{n,p} \leq m_{n,p+1}, \text{ because } N_p(x) \leq N_{p+1}(x),\quad p \in N .$$

Let us prove that for any $p \in N$ there exists $p'_2 \in N$ such that

$$\sum_{n\in Z} \frac{m_{n,p}}{m_{n,p'_2}} < \infty . \tag{6}$$

For large enough $n \in N$ we have

$$\frac{m_{n,p}}{m_{n,p_2}} = \frac{N_p(n)}{N_{p_2}(n)} \leq \frac{N_p(n)}{N_{p_2}(n+1)} \leq D_{p,p_2} \frac{N_p(n)}{N_{p_1}(n)}$$

where p_1 corresponds to p in (N) and p_2 is chosen in order to make $N_{p_1}(n) \leq N_{p_2}(n-1)$ hold. The existence of p_2 and D_{p_1,p_2} follows from (T). Namely, if (T) holds for $\{M_p(x)\}$ it is easy to prove that this condition holds for $\{N_p(x)\}$ as well. The con-

vergence of the series follows from (N), By the same arguments we obtain that

$$\sum_{n=-1}^{-\infty} \frac{m_{n,p}}{m_{n,p_2}} < \infty \quad , \text{ so (6) holds.}$$

Let us denote by $\Phi(E_n)$ the space of sequences of the form $\phi := =(\phi_0, \phi_1, \phi_{-1}, \phi_2, \phi_{-2} \ldots)$, where $\phi_0 \in E^{(c_{p,q})}_{[0,1]}$, $\phi_1 \in E^{(c_{p,q})}_{[1,2]}$, $\phi_{-1} \in E^{(c_{p,q})}_{[-1,0]}, \ldots,$ such that

$$\|\phi\|_{p,\phi} := \sum_{n\in Z} m_{n,p} \|\phi_n\|^2_{p,I} < \infty \quad , \ p \in N \ .$$

$(I_n = [n,n+1], \ n \in Z)$.

The proof that $\phi(E_n)$ is nuclear is the same as the proof of the nuclearity of the space $\phi(M)$ given in [2] p. 81. because the space $E_I^{(cp,q)}$ is countable Hilbert nuclear space according to the sequence of scalar products

$$(\phi,\psi)_p = \sum_{q=0}^{\infty} c_{p,q} \int_I \phi^{(q)}(x)\bar{\psi}^{(q)}(x)dx \ , \quad p \in N \ .$$

We embeded the space $\sigma\{N_p(x)c_{p,q}\}$ into the space $\Phi(E_n)$ by the isometry

$$i : \phi \to \{\phi|_{I_n} \ ; \ n = 0,1,-1,\ldots\}$$

The space $i(\sigma\{N_p(x)c_{p,q}\})$ is a closed subspace of $\Phi(E_n)$ and so it is nuclear.

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REZIME

NUKLEARNOST PROSTORA $\sigma\{M_p(x, q)\}$

U radu je pokazana nuklearnost prostora $\sigma\{M_p(x,q)\}$ gde je $M_p(x,q) = M_p(x)c_{p,q}$, a $\{M_p(x)\}$ i $\{c_{p,q}\}$ zadovoljavaju odredjene uslove.

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REMARKS ON DIFFERENT SPLITTINGS AND ASSOCIATED GENERALIZED LINEAR METHODS

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ABSTRACT

In this paper we consider some iterative methods for a linear system of equations Ax=b and their connection with the generalized linear method of the Newton-SOR and SOR-Newton type, [12]. Some sufficient conditions for the convergence of the linear method and for the local convergence of the generalized linear method are given.

INTRODUCTION

We shall consider a system of n linear equations with n unknowns, written in a matrix form

$$Ax = b,$$

where A is a nonsingular matrix with nonvanishing diagonal elements. One of the basic principles used in the generation and analysis of the iterative method for linear equations is splitting. That is, for the linear system Ax=b the matrix A is decomposed, or split, into the sum

$$A = B - C$$

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of two matrices, where B is nonsingular and the linear system $Bx=d$ is easy to solve. Then an iterative method is defined by

$$(1) \qquad x^{m+1} = B^{-1}Cx^m + B^{-1}b, \quad m=0,1,\ldots .$$

If we set $H = B^{-1}C$ then (1) can be written as

$$(2) \qquad x^{m+1} = x^o - (H^m + \ldots + E)B^{-1}(Ax^o - b) ,$$

where E is the unit matrix. Iteration (2) is convergent for all x^o if and only if the spectral radius $\rho(H)$ of matrix H is less than 1.

We shall now give some splittings of A and associated iterations of form (2). Let

$$A = D - T - S$$

be the decomposition of A into diagonal, strictly lower triangular, and strictly upper triangular parts. Let $F = \mathrm{diag}(f_1, \ldots \ldots, f_n)$ be a nonsingular matrix. Let for $\omega, \sigma \in \mathbb{R}$, $\omega \neq 0$,

$$B = \omega^{-1}(F-\sigma T), \quad C = \omega^{-1}(F-\omega D+(\omega-\sigma)T + \omega S) .$$

Matrix B is nonsingular, $A = B-C$ and

$$(3) \qquad B^{-1}C = (F-\sigma T)^{-1}(F-\omega D + (\omega-\sigma)T + \omega S) .$$

We denote by $H(F,\omega,\sigma)$ the matrix $B^{-1}C$ and by VAOR the associated iteration (2). If $F = D$ the VAOR reduces to AOR, [6]. Further, the AOR method for specific values of the parameters ω, σ reduces to well-known methods: for $\sigma = 0$, $\omega = 1$ the AOR method is the Jacobi method, for $\sigma = \omega = 1$ the AOR method is the Gauss-Seidel method, for $\sigma = 0$ the AOR method is the JOR method, for $\sigma = \omega$ the AOR method is the SOR method.

The sufficient conditions for the convergence of the AOR methods are considered by many authors including [1], [2], [6], [7], [9], [10], [11]. In this paper we shall give some sufficient conditions for the convergence of the VAOR method. Using these results we can also give some sufficient conditions for the local convergence of some generalized linear methods for the numerical solution of the system of nonlinear equations.

our results include some results from [5] and [12].

ON CONVERGENCE OF THE VAOR METHOD

We shall begin with some notations:
For $A = [a_{ij}] \in C^{n,n}$ (= set of complex $n \times n$ matrices) we define for $i=1,2,\dots,n$

$$P_i(A) = \sum_{j \in N(i)} |a_{ij}|, \quad Q_i(A) = \sum_{j \in N(i)} |a_{ji}|, \quad T_i = \max_{j \in N(i)} |a_{ji}| ,$$

where $N = \{1,2,\dots,n\}$, $N(i) = N \setminus \{i\}$.

THEOREM 1. *Let* $A = [a_{ij}] \in C^{n,n}$ *be such that*

(4) $$|a_{ii}||a_{jj}| > P_i(A)P_j(A), \quad i \in N, \quad j \in N(i) ,$$

or

(5) $$\alpha \in [0,1], \quad |a_{ii}| > P_i^{\alpha}(A)Q_i^{1-\alpha}(A), \quad i \in N ,$$

or

(6) $$\alpha \in [0,1], \quad |a_{ii}| > \alpha P_i(A) + (1-\alpha)Q_i(A), \quad i \in N .$$

Let $F = \operatorname{diag}(f_1,\dots,f_n) \in C^{n,n}$, *and let* $f_i/a_{ii} \in \mathbb{R}$, $f_i/a_{ii} > 0$, $i \in N$, $q = \min_{i \in N} \frac{f_i}{a_{ii}}$. *Then, for* $\omega \in (0,q]$, $\sigma \in [0,q]$, *we have* $\rho(H(F,\omega,\sigma)) < 1$.

P r o o f. The iteration matrix $H(F,\omega,\sigma)$ of the VAOR method is defined by (3). We assume that for some eigenvalue λ of $H(F,\omega,\sigma)$ $|\lambda| \geq 1$ holds. For this eigenvalue we have the following relation

$$\det(H(F,\omega,\sigma) - \lambda E) = 0.$$

Since $\det(F-\sigma T) = \det(F) \neq 0$, this is equivalent to $\det(Q) = 0$, where $Q = [q_{ij}]$ is defined by

$$Q = (F-\sigma T)(H(F,\omega,\sigma)-\lambda E) = (1-\lambda)F-\omega D+(\omega+\sigma(\lambda-1))T + \omega S .$$

In [6] it is proved that for $|\lambda| \geq 1$, $0 < x \leq 1$, $0 \leq y \leq 1$ we have

$$|\lambda-1+x| \geq |y(\lambda-1)+x|, \quad |\lambda-1+x| \geq x \ .$$

Now it follows that

$$|q_{ii}| = |(\lambda-1)f_i+\omega a_{ii}| = |\lambda-1+\omega \frac{a_{ii}}{f_i}||f_i| \geq \omega |a_{ii}| \ ,$$

and

$$|q_{ii}| \geq |(\lambda-1) \frac{\sigma a_{ii}}{f_i} + \omega \frac{a_{ii}}{f_i}||f_i| = |(\lambda-1)\sigma+\omega||a_{ii}| \ ,$$

since $0 < \omega a_{ii}/f_i \leq 1$, $0 \leq \sigma a_{ii}/f_i \leq 1$. It is easy to show that

$$P_i(Q) \leq |\frac{q_{ii}}{a_{ii}}| P_i(A) \ , \quad Q_i(Q) \leq |\frac{q_{ii}}{a_{ii}}| Q_i(A), \quad i \in N \ ,$$

and

$$P_i(Q)P_j(Q) < |q_{ii}||q_{jj}|, \quad i \in N, \ j \in N(i) \ ,$$

if (4) is true,

$$P_i^{\alpha}(Q)Q_i^{1-\alpha}(Q) < |q_{ii}| \ , \ i \in N \ ,$$

if (5) is true,

$$\alpha P_i(Q) + (1-\alpha)Q_i(Q) < |q_{ii}| \ , \quad i \in N$$

if (6) is true.

It follows now from [8], 2.4.1, 2.5.1, 2.5.2, that $\det Q \neq 0$. This contradicts the singularity of $H(F,\omega,\sigma)-\lambda E$. Therefore, $\rho(H(F,\omega,\sigma) < 1$.

COROLLARY. *Let* $A = [a_{ij}] \in C^{n,n}$ *be either strictly or irreducibly diagonally dominant. Let* $F = \mathrm{diag}(f_1 \ , \ldots, f_n) \in C^{n,n}$ *be nonsingular and let*

$$f_i/a_{ii} \in \mathbb{R}, \quad f_i/a_{ii} > 0, \quad i \in N, \quad q = \min_{i \in N} \frac{f_i}{a_{ii}} \ .$$

Then $\rho(H(F,\omega,\sigma)) < 1$ *for* $\omega \in (0,q]$, $\sigma \in [0,q]$.

P r o o f. We see that the eigenvalues of $H(F,\omega,\sigma)$ are the roots of $\det Q = 0$, with Q given in Theorem 1. With $|\lambda| \geq 1$, we know that Q is irreducible when A is irreducible. If A is (strictly) diagonally dominant, then Q is also a (strictly) diagonally dominant matrix. With these conditions, the value of λ such that $|\lambda| \geq 1$ can not be the eigenvalue of $H(F, \omega,\sigma)$ because Q is nonsingular.

THEOREM 2. *Let* $A = [a_{ij}]$, $F = \mathrm{diag}(f_1,\ldots,f_n) \in C^{n,n}$ *with* $a_{ii}, f_i > 0$, $i \in N$, *and let* $q = \min\limits_{i\in N} \frac{f_i}{a_{ii}}$. *Then* $\rho(H(F,\omega,\sigma)) < 1$ *for* $\omega \in (0,q]$, $\sigma \in [0,q]$ *if*

$$a_{ii} > \min(P_i(A), T_i(A)), \quad i \in N ,$$

$$a_{ii} + a_{jj} > P_i(A) + P_j(A) , \quad i \in N, \; j \in N(i) .$$

P r o o f. As in the proof of Theorem 1 the eigenvalues of the matrix $H(F,\omega,\sigma)$ are the roots of $\det Q = 0$. With $|\lambda| \geq 1$ we have

$$T_i(Q) \leq \frac{|q_{ii}|}{a_{ii}} T_i(A) , \quad P_i(Q) \leq \frac{|q_{ii}|}{a_{ii}} P_i(A), \quad i \in N ,$$

and

$$\text{(7)} \qquad \min(P_i(Q), T_i(Q)) \leq \frac{|q_{ii}|}{a_{ii}} \min(P_i(A), Q_i(A)) < |q_{ii}| .$$

Further,

$$|q_{ii}| + |q_{jj}| \geq |q_{ii} + q_{jj}| = |(\lambda-1)(f_i+f_j) + \omega(a_{ii}+a_{jj})| .$$

Since,

$$\frac{f_i + f_j}{a_{ii}+a_{jj}} \geq \min\left(\frac{f_i}{a_{ii}}, \frac{f_j}{a_{jj}}\right) \geq q, \quad i \in N, \; j \in N(i) ,$$

we have, as in the proof of Theorem 1,

$$|q_{ii} + q_{jj}| \geq \omega(a_{ii} + a_{jj}) ,$$

$$|q_{ii} + q_{jj}| \geq |(\lambda-1)\sigma + \omega|(a_{ii} + a_{jj}) .$$

Now it holds for $i \in N$, $j \in N(i)$

$$(8) \quad P_i(Q) + P_j(Q) \leq |q_{ii}+q_{jj}| \; \frac{P_i(A)+P_j(A)}{a_{ii}+a_{jj}} < |q_{ii}|+|q_{jj}| \; .$$

From (7) and (8) it follows by Theorem 5 from [13] that $\det Q = 0$, which contradicts the singularity of $H(F,\omega,\sigma)-\lambda E$.

ON SOME GENERALIZED LINEAR METHODS

In this section we shall consider the system of nonlinear equations $Gx = 0$, where $G: D \subset \mathbb{R}^n \to \mathbb{R}^n$, and suppose that G is F-differentiable and G' is continuous on an open neighbourhood $S_o \subset D$ of a point x^* for which $Gx^* = 0$.

One way to utilize the VAOR iteration in connection with nonlinear equations is to get approximate solutions of the linear systems which must be solved to carry out Newton's method. In this case, we would have a composite Newton-VAOR iteration, with Newton's method as the primary iteration and VAOR as the secondary iteration. In [12] it is shown that such a combination can be written in the form

$$(9) \quad x^{k+1} = x^k - (E+\ldots+H(x^k)^{m-1})B(x^k)^{-1}Gx^k, \; k=0,1,\ldots, \; m \geq 1 \; ,$$

where B and H are defined by

$$B(x) = \omega^{-1}(F(x) - \sigma T(x)) \; ,$$

$$C(x) = \omega^{-1}(F(x) - \omega D(x) + (\omega-\sigma)T(x) + \omega S(x)) \; ,$$

$$H(x) = B(x)^{-1}C(x), \quad \omega,\sigma \in \mathbb{R} \; , \; \omega \neq 0 \; .$$

In this case $F(x)$ is any nonsingular matrix and

$$G'(x) = D(x) - T(x) - S(x)$$

is the decomposition of $G'(x)$ into its diagonal, strictly lower, and strictly upper triangular parts, and it is assumed that $D(x)$ is nonsingular.

Under the above assumptions, x^* is a point of attraction of the iteration defined by (9) if $B: S_o \to \mathbb{R}^{n,n}$ is continuous

at x^*, $B(x^*)$ nonsingular and if $\rho(H(x^*)) < 1$, 10.3.1 from [12].

We can now use the results of the previous sections to obtain some sufficient conditions for a local convergence of the Newton-VAOR method, if we apply Theorem 1 and 2 to $H(x^*)$.

We have considered linear iterative methods in their traditional role of solving linear systems. However, it is also possible to give a direct extension of these methods to nonlinear equations, [12]. So, we have the one step SOR-Newton process and some of its modifications and generalizations [3], [4], [5], [12]. The one-step vSOR-Newton method, [5], is given by

$$x_i^{k+1} = x_i^k - \omega \frac{g_i(x^{k,i})}{f_i(x^{k,i})}, \quad i=1,\dots,n,$$

where, as usual, $g_1,\dots,g_n$ are the components of $G: D \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$\omega \in \mathbb{R}\setminus\{0\}, \quad x^{k,i} = [x_1^{k+1},\dots,x_{i-1}^{k+1}, x_i^k,\dots,x_n^k]^T, \quad i=1,\dots,n,$$

and functions $f_i: D \to \mathbb{R}$, $i=1,\dots,n$ are continuous on D with

$$f_i(x) > 0, \quad x \in D, \quad i=1,\dots,n.$$

For $f_i(x) = g_i'(x)$, $i=1,\dots,n$, this method reduces to the one-step SOR-Newton method.

In [3] a generalization of the vSOR-Newton method is given. This is the one-step vAOR-Newton method

$$x_i^{k+1} = x_i^k - \omega \frac{g_i(z^k)}{f_i(z^k)}, \quad i=1,\dots,n,$$

(10)

$$z_1^k = x_1^k - \frac{g_1(x^k)}{f_1(x^k)}, \quad z_i^k = x_i^k - \sigma \frac{g_i(z^{k,i})}{f_i(z^{k,i})}, \quad i=2,\dots,n,$$

where

$$\omega,\sigma \in \mathbb{R}\setminus\{0\}, \quad z^{k,i} = [z_1^k,\dots,z_{i-1}^k, x_i^k,\dots,x_n^k]^T, \quad i=1,\dots,n,$$

and $f_i: D \to \mathbb{R}$ are continuous on D and $f_i(x) > 0$ for $x \in D$, $i=1,\dots,n$.

This method reduces to the vSOR-Newton method if $\sigma = \omega$. Clearly, (10) may be written in the form $x^{k+1} = G_{\omega,\sigma}x^k$ although now the mapping $G_{\omega,\sigma}$ becomes rather complicated. In [3] it is shown that

$$G'_{\omega,\sigma}(x) = (F(x)-\sigma T(x))^{-1}(F(x)-\omega D(x) + (\omega-\sigma)T(x) + \omega S(x)) .$$

Now it is easy to see that $G'_{\omega,\sigma}(x) = H(x)$ with $F(x) = \text{diag}(f_1(x), \dots, f_n(x))$.

To prove the local convergence of the vAOR-Newton method it sufficies to show that $G_{\omega,\sigma}$ is F-differentiable at x^* and that $\rho(G'_{\omega,\sigma}(x^*)) < 1$, see the Ostrowski theorem, [12] . So, for the local convergence of the Newton-VAOR and vAOR-Newton method we consider the same matrix $H(x^*)$. We may apply the results of the previous section to $H(x^*)$ in order to obtain some sufficient conditions for the local convergence of these methods. In [3], [4] are given some sufficient conditions for the local convergence of the vAOR-Newton method. As special cases of this method we have the vSOR-Newton method from [5], and the SOR-Newton method from [12]. The Newton-VAOR method also contains, as a special case, the m-step Newton-SOR method ($m \geq 1$). Now, we can summarize our consideration of the system of nonlinear equations in the next theorem.

THEOREM 3. *Let* $G: D \subset \mathbb{R}^n \to \mathbb{R}^n$ *be F-differentiable in an open neighbourhood* $S_o \subset D$ *of a point* $x^* \in D$ *at which* G' *is continuous and* $Gx^* = 0$. *Let* $D(x)$, $-T(x)$, $-S(x)$ *be the diagonal strictly lower, and strictly upper triangular parts of* $G'(x)$. *Suppose that* $f_i(x) > 0$, $i=1,\dots,n$, *are continuous on* D *and* $F = \text{diag}(f_1(x),\dots,f_n(x))$. *If* $G'(x^*) = [g_{ij}]$ *and*

$$g_{ii} > 0,\ g_{ii}g_{jj} > P_i(G'(x^*))P_j(G'(x^*)),\ i \in N,\ j \in N(i) ,$$

or

(11) $\alpha \in [0,1]$, $g_{ii} > P_i^{\alpha}(G'(x^*))Q_i^{1-\alpha}(G'(x^*))$, $i \in N$,

or

$$\alpha \in [0,1], \quad g_{ii} > \alpha P_i(G'(x^*)) + (1-\alpha)Q_i(G'(x^*)), \quad i \in N,$$

or

$$g_{ii} > \min(P_i(G'(x^*)), T_i(G'(x^*)), \quad i \in N,$$

$$g_{ii}+g_{jj} > P_i(G'(x^*)) + P_j(G'(x^*)), \quad i \in N, \ j \in N(i),$$

or

$G'(x^*)$ *is irreducibly diagonally dominant,*

$G'(x^*)$ *is an M-matrix,*

then x^* *is a point of attraction of the Newton-VAOR and vAOR-Newton iteration for* $\sigma, \omega \in (0,q]$, *where* $q = \min_{i \in N} f_i(x^*)/g_{ii}$.

The proof of this theorem in the case that $G'(x^*)$ is an M-matrix, $0 < \sigma \le \omega \le q$ is given in [3]. If $G'(x^*)$ is strictly diagonally dominant, then (11) is true for $\alpha = 1$, and it follows that the statement of Theorem 3 also holds in this case.

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REZIME

PRIMEDBE NA RAZLIČITE DEKOMPOZICIJE I PRIDRUŽENE UOPŠTENE LINEARNE METODE

U radu se posmatraju neki iterativni postupci za rešavanje sistema linearnih jednačina Ax=b nastali dekomponovanjem matrice A u sumu A=B-C dve matrice, gde je B nesingularna matrica i takva da se sistem Bx=d može "lako rešiti". Formiran je VAOR iterativni postupak za iterativno rešavanje sistema Ax=b, koji kao specijalne slučajeve sadrži AOR, SOR i JOR postupke. Dati su neki dovoljni uslovi za konvergenciju VAOR postupka. Takodje se posmatraju kombinacije nelinearno-linearnih i linearno-nelinearnih postupaka za iterativno rešavanje sistema nelinearnih jednačina. Dati su neki dovoljni uslovi za lokalnu konvergenciju ovih postupaka. Kao specijalni slučajevi posmatranih postupaka javljaju se postupci Newton-SOR, SOR-Newton, [12], i vSOR-Newton, [5].

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ON A NUMERICAL SOLUTION OF A TYPE OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM BY USING A SPECIAL DISCRETIZATION MESH

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ABSTRACT

This paper presents a generalization of a mesh construction from [1] for a finite-difference discretization of a singularly perturbed problem (1). We give a class of functions that generate mesh points, enabling a quadratic convergence uniform in small perturbation parameter ε.

The possibilities of linear interpolation of numerical results is investigated as well, and the method is shown to be uniform in ε and to retain the accuracy order of numerical results.

1. INTRODUCTION

We consider the problem

(1a) $Tu := -\varepsilon^2 u'' + b(x,u) = 0, \qquad x \in I = [0,1],$

(1b) $Bu := (u(0),u(1)) = (U_0,U_1)$,

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under the basic assumptions:

$$b(x,u) \in C^k(I \times \mathbb{R}), \quad k \in \mathbb{N} \ ;$$

$$b_u(x,u) > \beta^2 > 0\ , \quad (x,u) \in I \times \mathbb{R}\ ; \quad 0 < \varepsilon \le \varepsilon_o\ ;$$

$$\beta, \quad \varepsilon_o, \quad U_o, \quad U_1 \in \mathbb{R}\ ,$$

where ε is a small perturbation parameter.

A problem of this type was considered, among the others, in [2] and the linear case of it in [1], [4], [5], [8]. It is well known that (T,B) is an inverse monotone operator and that there exists a unique solutions $u_\varepsilon \in C^{k+2}(I)$ to problem (1), see [2], [3]. The corresponding reduced problem

$$b(x,u) = 0\ , \quad x \in I,$$

also has a unique solution in $C^k(I)$, which, in general, does not satisfy the boundary conditions (1b). Therefore u_ε shows two boundary layers at the endpoints of the interval I.

We use a classical finite-difference scheme on a non-uniform mesh to solve (1) numerically. The discretization mesh is constructed in a special way, which generalizes the idea from [1], see [4] as well. This enables the second order convergence, uniform in ε, the result of which we shall state in section 4 and prove in section 5. To obtain this we have to know the estimates of u_ε and its derivatives and that is the subject of section 2.

In section 3, we shall give a general mesh construction, where the mesh points are obtained via $x_i = \lambda(i/n)$, $i = 0,1,\dots,n$, $n \in \mathbb{N}$, with some suitable function λ.

In section 6 we shall show that our discretization mesh is suitable to get the approximation of u_ε at any point $x \in I$ by interpolating numerical results. The linear interpolation retains the second order accuracy and informity in ε.

Section 7 contains some numerical results. They agree fully with the theoretical ones.

Throughout the paper M will denote each positive constant independent of ε and of the discretization mesh.

2. ESTIMATES OF u_ε AND ITS DERIVATIVES

Define the linear operator as:

$$L_o z := -\varepsilon^2 z'' + g_\varepsilon(x) z, \quad x \in I, \quad z \in C^2(I),$$

where

$$g_\varepsilon(x) = b(x,u_\varepsilon(x)) - b(x,0) = \int_0^1 b_u(x,su_\varepsilon(x))ds > \beta^2 > 0.$$

Obviously (L_o,B) is inverse monotone and we have

$$(2) \qquad L_o(\pm u_\varepsilon) = \mp b(x,0).$$

Now we can easily get:

LEMMA 1. *For the solution* u_ε *to problem* (1) *we have* $|u_\varepsilon^{(i)}(x)| \le M\varepsilon^{-i}$, $i = 0,1,\ldots,k+2$, $x \in I$.

P r o o f. For $i = 0$ the proof follows immediately from (2). For $i = 2$ we get the desired inequality directly from (1a) and for $i = 1$ we can use Lemma 1 from [1]. Further inequalities can be proved by differentiating (1a). We just have to use the formula for differentiating $b(x,u(x))$ from [2], page 35. □

LEMMA 2. *For the solution* u_ε *to problem* (1) *the following estimates hold:*

$$(3) \quad |u_\varepsilon^{(i)}(x)| \le M(1 + \varepsilon^{-i}V_\varepsilon(x)), \; i = 1,\ldots,k, \quad x \in I,$$

where $V_\varepsilon(x) = v_\varepsilon(x) + w_\varepsilon(x)$,

$$v_\varepsilon(x) = \exp(-\beta x/\varepsilon), \qquad w_\varepsilon(x) = \exp(-\beta(1-x)/\varepsilon).$$

P r o o f. For $z \in C^2(I)$ we take

$$Lz = -\varepsilon^2 z'' + b_u(x,u_\varepsilon(x))\cdot z \quad .$$

Then:

$$L(\pm u_\varepsilon') = \mp b_x(x,u_\varepsilon) \quad .$$

Because of the inverse monotonicity of (L,B) we can get (3) for $i = 1$. Here we use $|u_\varepsilon'(s)| \le M/\varepsilon$, $s = 0,1$, from Lemma 1.

Now suppose that (3) holds for $i = 1,2,\dots,j-1$, $2 \le j \le k$. We shall prove (3) for $i = j$. Consider

$$(4) \quad L(\pm u_\varepsilon^{(j)}) = \mp((b(x,u_\varepsilon))^{(j)} - b_u(x,u_\varepsilon)\cdot u_\varepsilon^{(j)})$$

and use the already mentioned formula from [2]. We get

$$L(\pm u_\varepsilon^{(j)}) \le M(1 + \varepsilon^{-j}V_\varepsilon) \quad .$$

We could use the inductive hypothesis since on the right hand side of (4) we have derivatives of u_ε up to the order $j-1$. The proof now follows from the inverse monotonicity of (L,B).□

The following theorem is proved in [4] in the linear case.

THEOREM 1. The solution u_ε to problem (1) can be represented in the following way:

$$u_\varepsilon = m + y_\varepsilon \ ,$$

where for $i = 0,1,\dots,k$ *and* $x \in I$ *we have*

$$(5) \qquad |m^{(i)}(x)| \le M \quad ,$$

$$(6) \qquad |y_\varepsilon^{(i)}(x)| \le M\varepsilon^{-i}V_\varepsilon(x) \ .$$

P r o o f. Consider the operator L_o. We can extend $g_\varepsilon(x)$ to the interval $[-1,2]$ in such a way that the smoothness and the property $g_\varepsilon(x) > \beta^2$ still hold. Denote this extension by $\overline{g}_\varepsilon(x)$. In the same way we make the extension $\overline{b}(x,0)$ of $b(x,0)$.

Let $m(x)$ be the unique solution to the problem

$$-\varepsilon^2 m'' + \bar{g}_\varepsilon(x)m = -\bar{b}(x,0), \quad x \in [-1,2],$$
$$m(-1) = m(2) = 0 .$$

Then (5) is obvious.

Now $y_\varepsilon = u_\varepsilon - m$ and we have

$$L_o y_\varepsilon = 0, \quad x \in I, \quad y_\varepsilon(s) = U_s - m(s), \quad s = 0,1 .$$

From the inverse monotonicity of (L_o, B) we get (6) for $i = 0$. Suppose that (6) holds for all $i = 0,1,\dots,j-1$, $i \le j \le k$. We have

$$L_o(\pm y_\varepsilon^{(j)}) = \mp((g_\varepsilon(x)y_\varepsilon)^{(j)} - g_\varepsilon(x)y_\varepsilon^{(j)}) \quad .$$

Because of Lemma 2 it follows

$$|g_\varepsilon^{(i)}(x)| \le M(1 + \varepsilon^{-i}V_\varepsilon(x)), \quad i = 0,1,\dots,j, \quad x \in I$$

and

$$L_o(\pm y_\varepsilon^{(j)}) \le M\varepsilon^{-j}V_\varepsilon \quad ,$$

so, we can prove (6) for $i = j$ in the same way as we have proved (3) in Lemma 2. □

3. MESH CONSTRUCTION

From now on we shall take $k = 4$.

Let $q \in (0,1/2)$. Consider the function $\phi \in C^3\ [0,q)$ with the properties

$$\phi^{(i)}(t) > 0, \quad i = 0,1,2,3, \qquad t \in (0,q)$$
$$\phi(0) = 0, \quad \phi(q) = +\infty,$$

and

$$\mu(t) := \phi'(t)\exp(-\phi(t)) \in C^2\ [0,q].$$

Let $A(t) = \int_t^q \mu(s)ds$, $t \in [0,q]$. We have

$$\phi(t) = -\ln A(t), \quad t \in [0,q)$$

and

(7) $\phi^{(i)}(t) \leq MA(t)^{-i}, \quad i = 1,2,3, \quad t \in [0,q).$

The examples for such a function are:

$$\phi_o(t) = -\ln(1 - (t/q)^p), \quad \text{for } p = 1,2, \text{ or } p \in [3,\infty);$$

and

$$\phi_1(t) = (q/(q-t))^p - 1, \quad \text{for } p > 0.$$

Let $\psi(t) = a\varepsilon\phi(t)$, $t \in [0,q)$, where $a\beta \geq 2$ and suppose $a\varepsilon_o\phi'(0) < 1$. Then $\psi'(0) < 1$ and there exists a unique point $\alpha \in (0,q)$ at which $\psi(t)$ contacts its tangent line from $(1/2,1/2)$.

Let

$$\psi'(\alpha_1) = 1/(1-2q), \quad \psi'(\alpha_2) = 1 .$$

The points α_1 and α_2 exist uniquely and we have $0 < \alpha_2 < \alpha < \alpha_1 < q$.

Take

$$\lambda(t) = \begin{cases} \psi(t), & t \in [0,\alpha] \\ \psi(\alpha) + \psi'(\alpha)(t-\alpha), & t \in [\alpha,1/2] \\ 1 - \lambda(1-t), & t \in [1/2,1] \end{cases}$$

We construct the mesh points x_i by

(8) $x_i = \lambda(t), \quad t_i = i/n, \quad i = 0,1,\ldots,n,$
$n = 2n_o, \quad n_o \in \mathbb{N}$.

To use $\lambda(t)$ we have to know α. It is the solution to the equation

(9) $$\psi(\alpha) + \psi'(\alpha)(1/2 - \alpha) = 1/2$$

which can be solved by successive approximations as in [1]. Note that for ϕ_1 with $p = 1$ (9) reduces to a quadratic equation and α can be easily evaluated.

For $p = 1$ ϕ_o is the function from [1]. The function ϕ_1 for $p \in \mathbb{N}$ is more convenient for practical use because it is a simple rational function.

4. DISCRETIZATION OF (1) AND THE CONVERGENCE THEOREM

Let $h_i = x_i - x_{i-1}$, $i = 1,2,\dots,n$. We form the discretization of problem (1):

$$u_o = U_o$$

(10) $$T_h u_i := -\varepsilon^2 D_h u_i + b(x_i, u_i) = 0, \quad i = 1,2,\dots,n-1,$$

$$u_n = U_1,$$

where

$$D_h u_i = \frac{2}{(h_i + h_{i+1})h_i h_{i+1}}(h_{i+1}u_{i-1} - (h_i + h_{i+1})u_i + h_i u_{i+1}).$$

The solution $u_h = [u_o, u_1, \dots, u_n]^T \in \mathbb{R}^{n+1}$ to the non-linear system (10) exists uniquely and it can be evaluated by the Newton method, see [7] for instance. Note that the perturbation parameter causes no truoble in the convergence of this method.

The system (10) can be written in the form:

$$A_h u_h + B_h u_h = f_h \quad ,$$

where $f_h = [U_o, 0, \dots, 0, U_n]^T \in \mathbb{R}^{n+1}$; $A_h = [a_{ij}] \in \mathbb{R}^{n+1,n+1}$ is a tridiagonal matrix with elements:

$$a_{oo} = a_{nn} = 1$$

and for $i = 1,2,\dots,n-1$

$$a_{i,i-1} = -2\varepsilon^2/((h_i + h_{i+1})h_i) \ ;$$

$$a_{ii} = 2\varepsilon^2/(h_i h_{i+1}) ,$$

$$a_{i,i+1} = - 2\varepsilon^2/((h_i + h_{i+1})h_{i+1}) ;$$

and $B_h u_h = \mathrm{diag}(0,b(x_1,u_1),\dots,b(x_{n-1},u_{n-1}),0) \in \mathbb{R}^{n+1,n+1}$. Putting $u_\varepsilon^h = [u_\varepsilon(x_o),u_\varepsilon(x_1),\dots,u_\varepsilon(x_n)]^T \in \mathbb{R}^{n+1}$ and $r_h = [0,r_1,r_2,\dots,r_{n-1},0]^T \in \mathbb{R}^{n+1}$, where

$$r_i = r_i(u_\varepsilon) = (Tu_\varepsilon)(x_i) - T_h u_\varepsilon(x_i) =$$
$$= \varepsilon^2(D_h u_\varepsilon(x_i) - u''_\varepsilon(x_i)), \quad i = 1,2,\dots,n-1,$$

we can easily get, see [6]:

(11) $$\|u^h - u_h\| \le \frac{1}{\beta}\|r_h\| .$$

Here $\|\cdot\|$ denotes the maximum norm: $\|z_h\| = \max_{0\le i\le n} |z_i|$ for $z_h = [z_o,z_1,\dots,z_n]^T \in \mathbb{R}^{n+1}$.

Thus, for our discretization (10) we have a stability uniform in ε, (11).

In the next section we shall prove the following theorem (a second order consistency, uniform in ε):

THEOREM 2. *Let the mesh points be given by* (8) *and let* $a\beta \ge 2$, $a\varepsilon_o\phi'(0) < 1$, $n > 3/q$ *and* $k = 4$. *Then we have*

$$\|r_h\| \le M/n^2 .$$

From this and (11) we get immediately

THEOREM 3. *Under the assumption of the previous theorem we have*

$$\|u_\varepsilon^h - u_h\| \le M/n^2 .$$

5. PROOF OF THE CONSISTENCY THEOREM

To prove Theorem 2 we use the same technique as in Theorem 3 from [1].

First we have $r_i(u_\varepsilon) = r_i(m) + r_i(y_\varepsilon)$, $i = 1,2,\dots,n-1$, and since $|r_i(m)| \le M/n^2$, we only have to prove

(12) $$|r_i(v_\varepsilon)| \le M/n^2, \quad i = 1,2,\dots,n_o-1 ,$$

because for $i = n_o, n_o+1,\dots,n-1$ and w_ε the proof of (12) is analogous.

Now let $r_i = r_i(v_\varepsilon)$. We have

(13) $$|r_i| \le \varepsilon^2 \tfrac{1}{3}(h_{i+1} - h_i)|v_\varepsilon'''(x_i)| + \varepsilon^2 \tfrac{1}{6}h_{i+1}^2|v_\varepsilon^{iv}(\theta_i)|$$

and

(14) $$|r_i| \le \varepsilon^2 \cdot 2|v_\varepsilon''(\eta_i)|,$$

with $\theta_i, \eta_i \in (x_{i-1}, x_{i+1})$. Using the definition of mesh points and the estimates from Theorem 1 we get from (13)

(15a) $$|r_i| \le M(P_i + Q_i)/n^2 \quad ,$$

(15b) $$P_i = \lambda''(t_{i+1}) \tfrac{1}{\varepsilon}v_\varepsilon(x_i) \quad ,$$

(15c) $$Q_i = (\lambda'(t_{i+1}))^2\varepsilon^{-2}v_\varepsilon(x_{i-1}) \quad ;$$

and from (14)

(16) $$|r_i| \le Mv_\varepsilon(x_{i-1}) .$$

For the function $\lambda(t)$, $t \in [0,1]$, we have

(17) $$\lambda'(t) \le 1/(1 - 2q) \quad ,$$
$$|\lambda''(t)| \le a\varepsilon\phi''(\alpha_1)$$

and because of (7)

(18) $$|\lambda''(t)| \le M\varepsilon A(\alpha_1)^{-2} = M\varepsilon(\phi'(\alpha_1)/\mu(\alpha_1))^2 \le M/\varepsilon .$$

1° Let $t_{i-1} \geq \alpha_2$. Then

$$v_\varepsilon(x_{i-1}) \leq v(\lambda(\alpha_2)) = \exp(-a\beta\phi(\alpha_2)) \leq \exp(-2\phi(\alpha_2)) =$$
$$= (\mu(\alpha_2)/\phi'(\alpha_2))^2 \leq M\varepsilon^2 .$$

Using this inequality and (18) from (15b) we get $P_i \leq M$. From (15c) and (17) we get $Q_i \leq M$ in this case. Thus (15) gives us (12).

2° Now let $t_{i-1} < \alpha_2$ and $t_{i-1} \leq q - 3/n$. Then $t_{i+1} \leq q - 1/n < q$ and

(19) $$q - t_{i+1} \geq \frac{1}{3}(q - t_{i-1}) \quad .$$

Because of

$$\lambda''(t_{i+1}) \leq \psi''(t_{i+1}) \quad ,$$

from (15b) we get

$$P_i \leq M\phi''(t_{i+1})\exp(-2\phi(t_{i-1})) \leq$$
$$\leq M(A(t_{i-1})/A(t_{i+1}))^2$$

and because of (19) $P_i \leq M$.

In the same way we use $\lambda'(t_{i+1}) \leq \psi'(t_{i+1})$ to get $Q_i \leq M$ from (15c). Then from (15) we have (12) in this case.

3° The last case is $q - 3/n < t_{i-1} < \alpha_2$. Note that $q - 3/n > 0$. Now it follows

$$\exp(-2\phi(t_{i-1})) < \exp(-2\phi(q - \tfrac{3}{n})) =$$
$$= A(q - \tfrac{3}{n})^2 \leq M/n^2$$

and from (16) we conclude (12) in this case and the theorem is proved.

6. LINEAR INTERPOLATION

For any $[z_0, z_1, \ldots, z_n]^T \in \mathbb{R}^{n+1}$ and $x \in [x_i, x_{i+1}]$, $i = 0,1,\ldots,n-1$, let

$$\ell(z_i, x) = z_i + \frac{1}{h_{i+1}}(z_{i+1} - z_i)(x - x_i) \quad .$$

We aproximate $u_\varepsilon(x)$, $x \in [x_i, x_{i+1}]$, by $\ell(u_i, x)$, where, as before, u_i denotes the solution to the discrete problem (10) on the mesh (8).

THEOREM 4. *Under the assumptions of Theorem 2 we have*

$$|u_\varepsilon(x) - \ell(u_i, x)| \le M/n^2 \;, \quad x \in [x_i, x_{i+1}] \;.$$

P r o o f. Let $x \in [x_i, x_{i+1}]$. Because of Theorem 2 we have

$$|\ell(u_\varepsilon(x_i), x) - \ell(u_i, x)| \le M/n^2 \;.$$

Now we shall prove

$$|u_\varepsilon(x) - \ell(u_\varepsilon(x_i), x)| \le M/n^2 \;.$$

Again, it is sufficient to show that

$$|R_i| \le M/n^2 \;, \quad i = 0,1,\ldots,n_o-1 \;, \tag{20}$$

where $R_i = v_\varepsilon(x) - \ell(v_\varepsilon(x_i), x)$. For other i's the proof of (20) is analogous. We have

$$|R_i| \le M(\lambda'(t_{i+1}))^2 \varepsilon^{-2} v_\varepsilon(x_i)/n^2$$

and we get (20) in both cases 1^o and 2^o of the proof of Theorem 2. In case 3^o we use

$$|R_i| \le M v_\varepsilon(x_i)$$

to get (20) again. □

7. NUMERICAL RESULTS

We shall test our method on the following linear problem from [8]:

$$(21)\quad \begin{aligned} &-\varepsilon^2 u''(x) + u(x) = -(\cos^2 \pi x + 2(\varepsilon\pi)^2 \cos 2\pi x) ,\\ &x \in I, \quad u(0) = u(1) = 0 , \end{aligned}$$

with the exact solution:

$$u_\varepsilon(x) = (\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon))/(1 + \exp(-1/\varepsilon)) - \cos^2 \pi x .$$

Since $u_\varepsilon(1/2 + x) = u_\varepsilon(1/2 - x)$, $x \in [0,1/2]$, it is sufficient to solve (21) on the interval $[0,1/2]$.

We use the mesh given via ϕ_1 with $p = 1$, because it is the simplest function and the results for ϕ_o and ϕ_1 with $p \neq 1$ are very similar. Note that here $\alpha_2 = q - \sqrt{aq\varepsilon}$, and we do not need the condition $a\beta \geq 2$ in Theorem 2. So, a is such a constant that $0 < a\varepsilon_o/q < 1$.

In our numerical experiments we shall vary ε, a, q and n_o. The width of the boundary layer is of order ε. We shall be interested in a number n_1 of mesh points in $(0,\varepsilon]$. For a,q and n_o fixed, this number is invariable to the change of ε. Let

$$E = \max_{n_1 < i < n_o} |u_\varepsilon(x_i) - u_i| ,$$

$$E_1 = \max_{0 < i \leq n_1} |u_\varepsilon(x_i) - u_i|$$

and let P and P_1 be the corresponding maximal percentage errors.

Tables 1-4 contain the results for u_i. In Table 5 we give the results of linear interpolation. We interpolate the numerical results of the first row of Table 4.

TABLE 1. $a = 1,\ q = 0.4,\ n_o = 10 \Rightarrow n_1 = 4$

ε	E_1	E	P_1	P
0.1	$7.22 \cdot 10^{-3}$	$3.14 \cdot 10^{-3}$	1.3	4.4
$10^{-2}-10^{-16}$ *)	$1.35 \cdot 10^{-2}$	$1.72 \cdot 10^{-2}$	2.2	2.1

*) ε was changed as $\varepsilon = 10^{-2s}$, $s = 1,2,\dots,8$.

TABLE 2. $a = 0.5,\ q = 0.48,\ n_o = 10 \Rightarrow n_1 = 6$

ε	E_1	E	P_1	P
0.1	$1.62 \cdot 10^{-2}$	$2.21 \cdot 10^{-2}$	3.3	3.9
$10^{-2}-10^{-16}$	$1.74 \cdot 10^{-2}$	$3.31 \cdot 10^{-2}$	3.1	3.6

TABLE 3. $a = 0.5,\ q = 0.48,\ n_o = 20 \Rightarrow n_1 = 12$

ε	E_1	E	P_1	P
0.1	$4.15 \cdot 10^{-3}$	$5.70 \cdot 10^{-3}$	0.83	3.7
$10^{-2}-10^{-16}$	$4.20 \cdot 10^{-3}$	$7.24 \cdot 10^{-3}$	0.74	0.87

TABLE 4. $n_o = 100$, $\varepsilon = 10^{-6}$

	n_1	E_1	E	P_1	P
$a = 1$, $q = 0.4$	40	$1.32 \cdot 10^{-4}$	$1.71 \cdot 10^{-4}$	0.021	0.021
$a = 0.3$, $q = 0.49$	75	$3.78 \cdot 10^{-4}$	$6.70 \cdot 10^{-4}$	0.061	0.074

TABLE 5. $a = 1,\ q = 0.4,\ n_o = 100,\ \varepsilon = 10^{-6}$

x	$E_2 = \lvert \ell(u_i, x) - u_\varepsilon(x) \rvert$	$(E_2/\lvert u_\varepsilon(x)\rvert)\cdot 100$
10^{-9}	$5.62\cdot 10^{-6}$	0.56
10^{-7}	$3.17\cdot 10^{-6}$	0.0033
10^{-3}	$1.65\cdot 10^{-5}$	0.0017
0.1	$4.19\cdot 10^{-4}$	0.046
0.2	$1.36\cdot 10^{-4}$	0.021
0.3	$6.96\cdot 10^{-5}$	0.020
0.4	$1.05\cdot 10^{-4}$	0.11
0.45	$6.38\cdot 10^{-5}$	0.26

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REZIME

O NUMERIČKOM REŠAVANJU JEDNOG TIPA SINGULARNO PERTURBIRANOG PROBLEMA KORIŠĆENJEM SPECIJALNE MREŽE DISKRETIZACIJE

U radu se daje uopštenje konstrukcije mreže iz [1] za diskretizaciju singularno preturbiranog problema (1) metodom konačnih razlika. Nalazi se klasa funkcija koje generišu tačke mreže, omogućujući kvadratnu konvergenciju, uniformnu po malom perturbacionom parametru ε. Takodje su ispitane mogućnosti linearne interpolacije numeričkih rezultata i za ovaj metod je pokazana uniformnost po ε i očuvanje reda tačnosti numeričkih rezultata.

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ON A LOCAL CONVERGENCE OF THE vAORN METHOD

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ABSTRACT

In this paper we consider a method for the numerical solution of nonlinear systems of equations. The method is a two-parameter generalization of the vSOR-Newton method (vSORN). When the two parameters involved are equal, it coincides with the vSORN method from [1] as a special case. This method we call vAORN ("verallgemeinerte" Accelerated Overrelaxation Newton) method.

1. INTRODUCTION

We shall consider the system of nonlinear equations

$$Fx = 0,$$

where

$$F: S \subset \mathbb{R}^n \to \mathbb{R}^n .$$

For some $x^0 \in S$ and some $\sigma, \omega \in \mathbb{R}$, $\omega \neq 0$, the iterates $\{x^k\}$ are defined by

(vAORN) $$x_i^{k+1} = x_i^k - \omega \frac{F_i(z^k)}{d_i(z^k)}, \quad i=1,2,\dots,n; \; k=0,1,\dots,$$

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where $z_1^k = x_1^k - \sigma \dfrac{F_1(x^k)}{d_1(x^k)}$, $z_i^k = x_i^k - \sigma \dfrac{F_i(z^{k,i})}{d_i(z^{k,i})}$, $i=2,3,\dots,n$,

$$z^{k,i} = [z_1^k,\dots,z_{i-1}^k, x_i^k,\dots,x_n^k]^T ,$$

and $d_i : S \to \mathbb{R}$, $i=1,2,\dots,n$.

We assume that:

1) F is F-differentiable on an open neighborhood $S_o \subset S$ of a point x^*, for which $Fx^* = 0$.

2) The functions d_i, $i=1,2,\dots,n$ are continuous on S and $d_i(x) > 0$, $i=1,2,\dots,n$, $x \in S$.

3) $f_i = \dfrac{\partial F_i}{\partial x_i}(x^*) \neq 0$, $i=1,2,\dots,n$, and without any restriction of the generality we can suppose that $f_i > 0$, $i=1,2,\dots,n$. Under these assumptions we shall prove the local convergence of the vAORN method using the theorem of Ostrowski, [3].

In case that $\sigma=\omega$ the vAORN method reduces to the vSORN method from [1]. In this case, if $F'(x^*)$ is a strictly diagonally dominant matrix, we get the convergence interval I_o for ω wider than the one from [1]. For $\sigma \in I_o$, using Theorem 1 from [4], we get a narrower convergence interval for ω than in [4]. In case that $d_i(x) = \dfrac{\partial F_i}{\partial x_i}(x)$, $i=1,2,\dots,n$ and $Fx = Ax + b$, where $A \in \mathbb{R}^{n,n}$ (= set of real $n \times n$ matrices) and $b \in \mathbb{R}^n$, the vAORN method is the AOR method from [2].

Let $G_{\sigma,\omega}$ be an iteration function for (vAORN) and let $F'(x^*) = D_F - L_F - U_F$ be the decomposition of $F'(x^*)$ into its diagonal, strictly lower, and strictly upper triangular parts. Let $D = \mathrm{diag}(d_1(x^*), d_2(x^*), \dots, d_n(x^*))$. For $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $\alpha \in [0,1]$ we define for $i=1,2,\dots,n$

$$P_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad Q_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}|,$$

$$P_{i,\alpha}(A) = \alpha P_i(A) + (1-\alpha) Q_i(A) .$$

2. THE LOCAL CONVERGENCE OF THE vAORN METHOD

Let $\sigma \neq 0$ and let G_σ be an iteration function for the vSORN method

$$\text{(vSORN)} \quad x_i^{k+1} = x_i^k - \sigma \frac{F_i(x^{k,i})}{d_i(x^{k,i})}, \quad i=1,2,\dots,n; \quad k=0,1,\dots$$

with $\quad x^{k,i} = [x_1^{k+1},\dots,x_{i-1}^{k+1},x_i^k,\dots,x_n^k]^T$,

from [1]. Then $G_{\sigma,\omega} = (1-\frac{\omega}{\sigma})E + \frac{\omega}{\sigma}G_\sigma$,

Since G_σ is F-differentiable at x^* (Theorem 1 from [1]), $G_{\sigma,\omega}$ is also F-differentiable at the same point and

$$\text{(1)} \quad G'_{\sigma,\omega}(x^*) = (D-\sigma L_F)^{-1}(D-\omega D_F + (\omega-\sigma)L_F + \omega U_F) .$$

For $\sigma = 0$, $G_{0,\omega}(x) = x-\omega D^{-1}(x)F(x)$ and $G'_{0,\omega}(x^*) = E - \omega D^{-1}F'(x^*)$, which is a special case of (1) for $\sigma = 0$. Thus, (1) is true for $\sigma,\omega \in \mathbb{R}$, $\omega \neq 0$.

From now on we shall assume that the assumptions 1)-3) from the introduction are valid.

THEOREM 1. *Let* $\alpha \in [0,1]$ *and let* $d_i-|\sigma|P_{i,\alpha}(L_F) > 0$, $i=1,2,\dots,n$. *Then*

$$\rho(G'_{\sigma,\omega}(x^*)) \le \max_i \frac{|d_i-\omega f_i|+|\omega-\sigma|P_{i,\alpha}(L_F)+|\omega|P_{i,\alpha}(U_F)}{d_i-|\sigma|P_{i,\alpha}(L_F)} .$$

P r o o f. Let λ be any eigenvalue of $G'_{\sigma,\omega}(x^*)$ and suppose that

$$|\lambda| > \frac{|d_i-\omega f_i|+|\omega-\sigma|P_{i,\alpha}(L_F)+|\omega|P_{i,\alpha}(U_F)}{d_i-|\sigma|P_{i,\alpha}(L_F)}, \quad i=1,2,\dots,n .$$

After some manipulations we have

$$|a_{ii}| = |(\lambda-1)d_i+\omega f_i| \geq \alpha(|\omega+\sigma(\lambda-1)|P_i(L_F) + |\omega|P_i(U_F)) +$$

$$+ (1-\alpha)(|\omega+\sigma(\lambda-1)|Q_i(L_F)+|\omega|Q_i(U_F)) =$$

$$= \alpha P_i(A) + (1-\alpha)Q_i(A), \quad i=1,2,\dots,n ,$$

where $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $A = (\lambda-1)D+\omega D_F-(\omega+\sigma(\lambda-1))L_F-\omega U_F$. Then Theorem 2.5.2. from [6] shows that $\det A \neq 0$. Since $(D-\sigma L_F)(\lambda E - G'_{\sigma,\omega}(x^*)) = A$ and $\det(D-\sigma L_F) \neq 0$, it follows $\det(\lambda E-G'_{\sigma,\omega}(x^*)) \neq 0$. This contradicts the singularity of $\lambda E - G'_{\sigma,\omega}(x^*)$.

THEOREM 2. *Let for some* $\alpha \in [0,1]$, $f_i > P_{i,\alpha}(F'(x^*))$, $i=1,2,\dots,n$. *Then for*

$$0 < \omega < \min_i \frac{2d_i}{f_i+P_{i,\alpha}(F'(x^*))}$$

and

$$\max_i \frac{-\omega(f_i-P_{i,\alpha}(F'(x^*)))+2\max(0,\omega f_i-d_i)}{2P_{i,\alpha}(L_F)} <$$

$$< \sigma < \min_i \frac{\omega(f_i+P_{i,\alpha}(L_F)-P_{i,\alpha}(U_F))+2\min(0,d_i-\omega f_i)}{2P_{i,\alpha}(L_F)}$$

$\rho(G'_{\sigma,\omega}(x^*)) < 1$ *holds, i.e. the vAORN method converges locally.*

P r o o f. We shall prove that for all $i=1,2,\dots,n$, the following implication holds.

$$(2) \qquad \left.\begin{array}{c} 0 < \omega < \dfrac{2d_i}{f_i+P_{i,\alpha}(F'(x^*))} \\[2ex] \dfrac{-\omega(f_i-P_{i,\alpha}(F'(x^*)))+2\max(0,\omega f_i-d_i)}{2P_{i,\alpha}(L_F)} < \sigma < \\[2ex] < \dfrac{\omega(f_i+P_{i,\alpha}(L_F)-P_{i,\alpha}(U_F))+2\min(0,d_i-\omega f_i)}{2P_{i,\alpha}(L_F)} \end{array}\right\} \Rightarrow$$

$$(3) \qquad \frac{|d_i-\omega f_i|+|\omega-\sigma|P_{i,\alpha}(L_F)+|\omega|P_{i,\alpha}(U_F)}{d_i-|\sigma|P_{i,\alpha}(L_F)} < 1 .$$

Since for σ and ω from (2) we have $d_i - |\sigma| P_{i,\alpha}(L_F) > 0$, Theorem 1 and (3) show that $\rho(G'_{\sigma,\omega}(x^*)) < 1$.

Let us introduce the following notations: $\ell_i = P_{i,\alpha}(L_F)$, $u_i = P_{i,\alpha}(U_F)$.

To prove implications (2) => (3) we consider the next cases.

Case I: $0 < \omega \le \dfrac{d_i}{f_i}$, $\dfrac{-\omega(f_i - \ell_i - u_i)}{2\ell_i} < \sigma \le 0$.

Then $d_i - \omega f_i + \omega \ell_i - \sigma \ell_i + \omega u_i < d_i + \sigma \ell_i$.

Case II: $0 < \omega \le \dfrac{d_i}{f_i}$, $0 < \sigma \le \omega$.

Then $d_i - \omega f_i + \omega \ell_i - \sigma \ell_i + \omega u_i < d_i - \sigma \ell_i$, since $\ell_i + u_i < f_i$.

Case III: $0 < \omega \le \dfrac{d_i}{f_i}$, $\omega < \sigma < \dfrac{\omega(f_i + \ell_i - u_i)}{2\ell_i}$.

Then $d_i - \omega f_i + \sigma \ell_i - \omega \ell_i + \omega u_i < d_i - \sigma \ell_i$.

Case IV: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i + \ell_i + u_i}$, $\dfrac{\omega(f_i + \ell_i + u_i) - 2d_i}{2\ell_i} < \sigma \le 0$.

Then $\omega f_i - d_i + \omega \ell_i - \sigma \ell_i + \omega u_i < d_i + \sigma \ell_i$.

Case V: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i + \ell_i + u_i}$, $0 < \sigma \le \omega$.

Then $\omega f_i - d_i + \omega \ell_i - \sigma \ell_i + \omega u_i < d_i - \sigma \ell_i$.

Case VI: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i + \ell_i + u_i}$, $\omega < \sigma < \dfrac{\omega(-f_i + \ell_i - u_i) + 2d_i}{2\ell_i}$.

Then $\omega f_i - d_i + \sigma \ell_i - \omega \ell_i + \omega u_i < d_i - \sigma \ell_i$.

COROLLARY 2.1. *Let $F'(x^*)$ be a strictly diagonally dominant matrix. Then for*

$$0 < \omega < \min_i \frac{2d_i}{f_i + P_i(F'(x^*))} \quad \textit{and}$$

$$\max_i \frac{-\omega(f_i-P_i(F'(x^*)))+2\max(0,\omega f_i-d_i)}{2P_i(L_F)} < \sigma <$$

$$< \min_i \frac{\omega(f_i+P_i(L_F)-P_i(U_F))+2\min(0,d_i-\omega f_i)}{2P_i(L_F)}$$

$\rho(G'_{\sigma,\omega}(x^*)) < 1$ *holds, i.e. the vAORN method converges locally.*

The proof follows immediately from Theorem 2 with $\alpha=1$.

COROLLARY 2.2. *Let* $F'(x^*)$ *be a strictly diagonally dominant matrix. For the iteration function* G_ω *of the vSORN method the following implication holds*

$$0 < \omega < \min_i \frac{2d_i}{f_i+P_i(F'(x^*))} \Rightarrow \rho(G'_\omega(x^*)) < 1 .$$

P r o o f. For $\omega=\sigma$ we have $G'_{\omega,\omega}(x^*) = G'_\omega(x^*)$. Since for any $i=1,2,\ldots n$, $-\omega(f_i-P_i(F'(x^*)))+2\max(0,\omega f_i-d_i) < 0$ and

$$\omega < \frac{2d_i}{f_i+P_i(F'(x^*))} \Rightarrow \omega < \frac{\omega(f_i+P_i(L_F)-P_i(U_F))+2\min(0,d_i-\omega f_i)}{2P_i(L_F)},$$

using Theorem 2 we complete the proof.

REMARK 1. The local convergence of the vSORN method was proved in [1] for $\omega \in (0,q]$, where $q=\min_i \frac{d_i}{f_i}$ under the same assumptions as in Corollary 2.2. Our interval for ω is wider.

REMARK 2. Theorem 2 enables us to consider the local convergence of the vSORN method for a wider class of matrices than in [1].

REMARK 3. From Corollary 2.2. and Theorem 1 from [4] it follows that

$$0 < \omega \leq \sigma < \min_i \frac{2d_i}{f_i+P_i(F'(x^*))} \Rightarrow \rho(G'_{\sigma,\omega}(x^*)) < 1 .$$

Our Theorem 2 gives us more.

REMARK 4. For $d_i = \frac{\partial F_i}{\partial x_i}$, $Fx = Ax + b$, $A \in \mathbb{R}^{n,n}$, $b \in \mathbb{R}^n$, Theorem 2 is a special case of Theorem 3 from [5].

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REZIME

O LOKALNOJ KONVERGENCIJI VAORN POSTUPKA

U radu se posmatra postupak za numeričko rešavanje sistema nelinearnih jednačina Fx=0. Taj postupak, koji je dvoparametarska generalizacija vSOR-Njutnovog postupka (vSORN), razmatranog u [1], nazvali smo vAORN ("verallgemeinerte" Accelerated Overrelaxation Newton) postupak. Pod odredjenim pretpostavkama za funkciju F i matricu $F'(x^*)$, gde je x^* rešenje sistema Fx=0, odredjeni su intervali konvergencije za parametre σ i ω. U specijalnom slučaju, za $\sigma=\omega$ i kada je $F'(x^*)$ strogo dijagonalno dominantna matrica, interval konvergencije za ω dobijen u ovom radu širi je od odgovarajućeg iz [1]. Analogni rezultati iz ovog rada za slučaj sistema linearnih jednačina dati su u [5].

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THE SPECTRAL TYPE OF POLYNOMIALS OF THE GAUSSIAN PROCESS

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ABSTRACT

Let $\xi(t) = \int_0^t g(t,u)d\eta(u)$ be the proper canonical representation of the Gaussian process $\{\xi(t),\ t>0\}$ and let H_n be the linear closure of polynomials $P_n(\xi(t_1),\ldots,\xi(t_n))$. The conditional expectation $E_t(\cdot) = E(\cdot|\xi(u),\ u\leq t)$, $t\geq 0$, is a resolution of the identity in the separable Hilbert space H_n.

It is proved that the measure $||d\eta(u)||^2$ is the uniform maximal spectral type of the infinite multiplicity in H_n.

Let $\{\xi(t),\ t\geq 0\}$ be a Gaussian process with the proper canonical representation ([2]).

$$\xi(t) = \int_0^t g(t,u)d\eta(u),\ t\geq 0 \tag{1}$$

The process $\{\eta(u),\ u\geq 0\}$ is a martingal with

$$||\eta(t)||^2 = E\eta^2(t) = F(t) = \int_0^t f(u)du,\quad f(u)\geq 0 \quad \text{a.e.}$$

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Let $H^{(1)}(\xi)$ $(H_t^{(1)}(\xi))$ be the linear closure of $\{\xi(u),\ u \geq 0\}$ $(\{\xi(u),\ 0 \leq u \leq t\})$. It follows from (1) that $H_t^{(1)}(\xi) = H_t^{(1)}(\eta)$, $t \geq 0$. The linear time-domain analysis consists in the determination of the measure $dF(t)$. More precisely, the determination of the class of all measures equivalent (by absolute continuity) to $dF(t)$, ([1]).

The conditional expectation $E_t(\cdot) = E(\cdot\,|\,\xi(u),\ u \leq t)$ as the operator in $H^{(1)}(\xi)$ is the projection onto $H_t^{(1)}(\xi)$. So $\{E_t,\ t \geq 0\}$ is a resolution of the identity in the separable Hilbert space $H^{(1)}(\xi)$. The proper canonical representation (1) means that the space $H^{(1)}(\xi)$ is cyclic with the spectral type $dF(t)$, ([5]).

Now we shall consider the Hilbert space H_n, $(H_n(t))$ as the linear closure of all the polynomials (the degree not exceeding n) of the random variables $\{\xi(u),\ u \geq 0\}$, $(\{\xi(u),\ u \leq t\})$. The space H_n reduces $\{E(t),\ t \geq 0\}$, [6]. So $\{E_t,\ t \geq 0\}$ is a resolution of the identity in H_n. The non-linear time-domain analysis consists in the determination of the spectral type of $\{E_t,\ t \geq 0\}$ in H_n.

THEOREM. *The spectral type of* $\{E_t,\ t \geq 0\}$ *in* H_n *is*

$$dF \geq dF \geq \dots$$

In the terms of the spectral analysis of selfadjoint operators in the separable Hilbert space, ([5]), the theorem states that the spectral type of the cyclic subspace $H^{(1)}$ is the uniform maximal spectral type in H_n. The multiplicity of dF in H_n is infinite. It is proved in [3] that the spectral type of $\{E_t,\ t \geq 0\}$ in space H_n of the polynomials of the Wiener process $\{W(t),\ t \geq 0\}$ is $dt \geq dt \geq \dots$. The present theorem is a generalization of this result. The idea of the proof is the same as in [3], but the technique is more complicated.

P r o o f. The first step in the proof is the decomposition of H_n in the orthogonal sum of the subspaces $H^{(p)}$, $p=1,\dots,n$. $H^{(p)}$ ($H_t^{(p)}$) is the linear closure of the Hermite polynomials $H_p(t_1,\dots,t_p)=H_p(\xi(t_1),\dots,\xi(t_p))$, $0\le t_p\le \dots \le t_1$ of the degree p. We conclude by the relation ([4])

$$EH_p(\xi(t_1),\dots,\xi(t_p)) = H_p(E_t\xi(t_1),\dots,E_t\xi(t_p)),$$

that $H^{(p)}$ reduces $\{E_t,\ t\ge 0\}$. So it will be sufficient to prove that the spectral type of $\{E_t,\ t\ge 0\}$ in $H^{(p)}$, $p\ge 2$, is $dF(t)\ge dF(t)\ge \dots$.

In this way the theorem will be proved when we find the mutually orthogonal martingals $\{\eta_n(t),\ t\ge 0\}$, $n=1,2,\dots$ in $H^{(p)}$ such that

$$\text{(2)}\qquad \sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n) = H_t^{(p)}\ ,\ t\ge 0,$$

and

$$\text{(3)}\qquad ||\eta_n(t)||^2 = \int_0^t f_n(u)\,dF(u),\quad f_n(u)>0\ \text{ a.e. } dF\ .$$

We recall the fact, [6], that $H^{(p)}$ concides with the set $\{I_p\}$ of Ito-Rozanov integrals

$$I_p = \int_0^\infty \int_0^{t_1} \dots \int_0^{t_{p-1}} \phi(t_1,\dots,t_p)\,d\eta(t_1)\dots d\eta(t_p),\ .$$

$$||I_p||^2 = \int_0^\infty \int_0^{t_1} \dots \int_0^{t_{p-1}} \phi^2(t_1,\dots,t_p)\,dF(t_1)\dots dF(t_p).$$

Denote by $S_1(t)$ the section of $\Delta_1=\{(u_1,\dots,u_p): 0\le u_p\le \dots \le u_1\}\subset R_p$ at $u_1=t$ i.e. $S_1(t)=\{(t,u_2,\dots,u_p)\in\Delta_1\}$. The measure of $S_1(t)$ in R_{p-1} is $m(S_1(t))=\int_{S_1(t)} dF(u_2)\dots dF(u_p)$.

We partition Δ_1 into two subset Δ_2 and Δ_3 such that the measures of the corresponding section $S_2(t)$ and $S_3(t)$ are equal: $m(S_2(t))=m(S_3(t))=\frac{1}{2}m(S_1(t))$ for each t. Then partition Δ_2 into Δ_4 and Δ_5 such that $m(S_4(t))=m(S_5(t))=\frac{1}{2}m(S_2(t))$,

$t \geq 0$, partition Δ_3 into Δ_6 and Δ_7 such that $m(S_6(t))=m(S_7(t))= = \frac{1}{2}m(S_3(t))$ and so on. Let $I(t)$ be the support of the measure $dF(u_2)\ldots dF(u_p)$ on $S_1(t)$. We suppose that the diameter of $S_n(t) \cap I(t)$ tends to zero as $n \to \infty$, uniformly in t in each finite interval. One construction of Δ_n, $n=1,2,\ldots$ is done after the proof.

Let the partition of Δ_n be Δ_{n1} and Δ_{n2}. We define the processes $\{\eta_n(t),\ t \geq 0\}$, $n=1,2,\ldots$ by

$$\eta_1(t) = \int_0^t \{ \int_{S_1(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1) ,$$

$$\eta_n(t) = \int_0^t \{ \int_{S_{n1}(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1) - \int_0^t \{ \int_{S_{n2}(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1), \quad n=2,3,\ldots .$$

It is easy verify that $\{\eta_n(t)\}$, $n=1,2,\ldots$ are mutually orthogonal martingals. Also

$$||\eta_n(t)||^2 = \int_0^t \{ \int_{S_n(u_1)} dF(u_2)\ldots dF(u_p)\} dF(u_1) = \int_0^t m(S_n(u_1))\,dF(u_1) \quad \text{where } m(S_n(u)) > 0 \text{ a.s.}$$

with respect to $dF(u)$. Condition (3) is satisfied.

Consider the martingals $\{\zeta_{S_n}(t),\ t \geq 0\}$, $n=1,2,\ldots$

$$\zeta_{S_n}(t) = \int_0^t \{ \int_{S_n(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1).$$

It is easy to see that, for each t, $t \geq 0$, and n, $n=1,2,\ldots$, $\zeta_{S_n}(t)$ is the finite linear combination of $\eta_k(t)$, $k=1,2,\ldots$:

$$\zeta_{S_1}(t) = \eta_1(t),\quad \zeta_{S_2}(t) = \frac{1}{2}(\eta_1(t)+\eta_2(t)),$$

$$\zeta_{S_3}(t) = \frac{1}{2}(\eta_1(t)-\eta_2(t)),\ \ldots .$$

If $S^*(t) = \sum_{k=1}^{\ell} S_{j_k}(t)$, $t \geq 0$, where $S_{j_k}(t)$, $k=1,\dots,\ell$ are disjoint, then $\zeta_{S^*}(t) = \sum_{k=1}^{\ell} \zeta_{S_{j_k}}(t)$ or $\zeta_{S^*}(t) = \int_0^t \{\int_{S^*(u_1)} d\eta(u_2)\dots d\eta(u_p)\} d\eta(u_1)$.

Let $S(t)$ be a measurable subset of $S_1(t)$. We apply the standard limit procedure: if $S_m^*(t) \uparrow S(t)$ as $m \to \infty$, then

$\zeta_{S_m^*}(t) \to \zeta_S(t)$ or $\zeta_S(t) = \int_0^t \{\int_{S(u_1)} d\eta(u_2)\dots d\eta(u_p)\} d\eta(u_1)$, $t \geq 0$.

We conclude that

$$\zeta_S(t) \in \sum_{n=1}^{\infty} H_t^{(1)}(\eta_n), \quad t \geq 0.$$

Let D be a bounded measurable subset of Δ_1 and let $s = \inf\{u_1 : (u_1,\dots,u_p) \in D\}$, $t = \sup\{u_1 : (u_1,\dots,u_p) \in D\}$. Consider a partition $s = s_o < s_1 < \dots \; s_m = t$ of $[s,t]$. Denote by $\{\zeta_j(t), t \geq 0\}$ the martingal $\{\zeta_{S_{(j)}}(t), t \geq 0\}$ where $S_{(j)}(u_1)$ for $u_1 = s_j$ is the section of D at $u_1 = s_j$. Let

$$\xi_m = \sum_{j=o}^{m-1} \int_{s_j}^{s_{j+1}} d\zeta_j(u) = \sum_{j=o}^{m-1} [\zeta_j(s_{j+1}) - \zeta_j(s_j)]$$

It follows that $\xi_m \to \int_D d\eta(u_1)\dots d\eta(u_p)$, when $\max_{0 \leq j \leq m-1} (s_{j+1} - s_j) \to 0$.

So we have proved that

$$\int_D d\eta(u_1)\dots d\eta(u_p) \in \sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n) \text{ or}$$

$$\int_o^t \int_o^{u_1} \dots \int_o^{u_{p-1}} \phi(u_1,\dots,u_p) d\eta(u_1)\dots d\eta(u_p) \in$$

$$\sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n), \; t \geq 0.$$

This complets the proof.

<u>Construction of the sets</u> Δ_n, $n=2,3\dots$ First we consider the case $p=2$. Let $\phi_o(u) = 0$, $\phi_1(u) = u$, $u \geq 0$ and $\phi_2(u) = F^{-1}(\frac{1}{2}(F(\phi_o(u)) + F(\phi_1(u))) = F^{-1}(\frac{1}{2}F(u))$, $u \geq 0$. Remark that

$\phi_2(u)$, $u \geq 0$ is continuous and nondecreasing. Also $0 < \phi_2(u) < u$, $u \geq 0$. (Of course, we suppose that t=0 is the increasing point of F(t)). We may put $S_1(t) = [0,t]$, $S_2(t) = [0,\phi_2(t)]$, $S_3(t) = [\phi_2(t), 0]$. Indeed, $m(S_1(t)) = F(t)$, $m(S_2(t)) = \int_{S_2(t)} dF(u) = F(\phi_2(t)) = \frac{1}{2} F(t)$. Now let $\phi_3(u) = F^{-1}(\frac{1}{2}(F(\phi_o(u)) + F(\phi_2(u))))$. Partition $S_2(t)$ in $S_4(t) = [0,\phi_3(t)]$ and $S_5(t) = [\phi_3(t),\phi_2(t)]$. Generally, for $S_n(t) = [\phi_{n_1}(t),\phi_{n_2}(t)]$ let

$$\phi_{n*}(u) = F^{-1}\left(\frac{1}{2}(F(\phi_{n_1}(u)) + F(\phi_{n_2}(u)))\right), \quad u \geq 0 .$$

The function $\phi_{n*}(u)$, $u \geq 0$ is continuous and nondecreasing. Also, $\phi_{n_1}(u) < \phi_{n*}(u) < \phi_{n_2}(u)$, $u \geq 0$. The partitions of $S_n(t)$ are $S_{n'}(t) = [\phi_{n_1}(t), \phi_{n*}(t)]$ and $S_{n''}(t) = [\phi_{n*}(t),\phi_{n_2}(t)]$, because $m(S_{n'}(t)) = \int_{\phi_{n_1}(t)}^{\phi_{n*}(t)} dF(u) = F(\phi_{n*}(t)) - F(\phi_{n_1}(t)) = \frac{1}{2}(F(\phi_{n_2}(t)) - F(\phi_{n_1}(t)) = \frac{1}{2} m(S_n(t))$. Let p=3 and $S_1(t) = \{(t_1,t_2,t_3) \in \Delta_1\}$. Consider the subsets $S_1'(t) = \{(t ,t_2,t_3) : 0 \leq t_3 \leq t_2 \leq \frac{1}{2} t\}$ and $S_1''(t) = \{(t ,t_2,t_3) : 0 \leq t_3 \leq t_2, \frac{1}{2}t \leq t_2 \leq t\}$. Partition $S_1'(t)$ in $S_2'(t) = \{(t ,t_2,t_3) : 0 \leq t_3 \leq \phi_2(t_2), 0 \leq t_2 \leq \frac{1}{2}t\}$ and $S_3'(t) = \{(t, t_2,t_3) : \phi_2(t_2) \leq t_3 \leq t_2, 0 \leq t_2 \leq \frac{1}{2}t\}$. We have $m(S_2'(t)) = m(S_3'(t))$ $(= \frac{1}{4} F^2(\frac{1}{2}t))$, $t \geq 0$ because $m(S_2'(t)) = \int_0^{t/2}\int_0^{\phi_2(u_1)} dF(u_1)dF(u_2)$ and $m(S_3'(t)) = \int_0^{t/2}\int_{\phi_2(u_1)}^{u_1} dF(u_1)dF(u_2)$. Similarly, we partition $S_1''(t)$ in $S_4'' = \{(t ,t_2,t_3) : 0 \leq t_3 \leq \phi_2(t), \frac{1}{2}t \leq t_2 \leq t\}$ and $S_5''(t) = \{(t ,t_2,t_3) : \phi_2(t_2) \leq t_3 \leq t_2, \frac{1}{2}t \leq t_2 \leq t\}$. Define

$$\eta_1 = \int_o^t \int_{S_1}, \quad \eta_2 = \int_o^t \int_{S_2'} - \int_o^t \int_{S_3'}, \quad \eta_3 = \int_o^t \int_{S_4''} - \int_o^t \int_{S_5''} .$$

In the next step we partition $S_2'(t)$ $((S_3'(t))$ by $\frac{1}{4}t$ and $\phi_3(u)$ $(\phi_4(u))$. We partition the set $S_4''(t)$ $(S_5''(t))$ by $\frac{3}{4}t$ and $\phi_3(u)$ $(\phi_4(u))$ and so on. Define $\eta_4(\eta_5)$, η_6 (η_7), and so on.

Passing to the case p=4 we use the sets $S(t) = \{(t\ ,t_2,t_3,t_4) \in S^{(3)}(t_2)\}$ where $S^{(3)}(t_2)$ belongs to the sets involving the partitions in the case p=3. At the same time we partition $S_1(t) = \{(t\ ,t_2,t_3,t_4) \in \Delta_1\}$ in $S_1'(t) = \{0 \leq t_2 \leq \frac{1}{2}t\}$ and $S_1''(t) = \{\frac{1}{2}t \leq t_2 \leq t\}$ and so on. The procedure for arbitrary p, $p \geq 5$ follows by induction.

<u>A consequence</u>. Consider the Hilbert space H (H_t) of all random variables η, $E\eta = 0, E\eta^2 < +\infty$ measurable with respect to σ-field generated by $\{\xi(u),\ u \geq 0\}$ $(\{\xi(u),\ 0 \leq u \leq t\})$. Noting the relation $H_t = \sum_{p=0}^{\infty} \oplus H_t^{(p)}$, $t \geq 0$, ([6]), we have: the spectral type of $\{E_t,\ t \geq 0\}$ in H is $dF \geq dF \geq \dots$.

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REZIME

SPEKTRALNI TIP POLINOMA GAUSOVOG PROCESA

Neka je $\xi(t) = \int_0^t g(t,u)d\eta(u)$ čisto kanonička reprezentacija Gausovog procesa $\{\xi(t),\ t \geq 0\}$ i neka je H_n linearna zatvorenost polinoma $P_n(\xi(t_1),\dots,\xi(t_n))$. Uslovno očekivanje $E_t(.) = E_t(.|\xi\ (u),\ u \leq t)$ je razlaganje jedinice u separabilnom Hilbertovom prostoru H_n.

Dokazuje se da je mera $\|d\eta(u)\|^2$ uniformni maksimalni spektralni tip beskonačnog multipliciteta u H_n.

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A NOTE ON PRODUCT CURVATURE TENSORS

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ABSTRACT

The purpose of the present note is to find some relations connecting the product conformal curvature tensor, the product projective curvature tensor, the product concircular curvature tensor and the product conharmonic curvature tensor.

1. An n-dimensional differentiable manifold M_n of class C^∞ is called a locally decomposable Riemannian space [1] if in M_n a linear transformation field $F \neq I$ and a positive definite Riemannian metric g are given, satisfying

$$F^2 = I, \quad g(X,Y) = g(FX,FY), \quad (\nabla_X F)(Y) = 0$$

for any vector fields X and Y on M_n, where I denotes the identity transformation field and ∇ is the operator of the covariant derivative with respect to the Riemannian metric g. Putting

$$F(X,Y) = g(FX,Y) = g(X,FY)$$

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we have

$$F(X,Y) = F(Y,X).$$

The matrix (F^i_j) has ± 1 as the proper values. Let us denote by $T(x)$ the tangent vector space of M_n at a point p and let $P(x)$ and $Q(x)$ be the proper vector spaces corresponding to the proper values $+1$ and -1 respectively. If we put $\dim P(x) = p$, $\dim Q(x) = q$, then p and q are constants and it holds that $\phi = F^i_i = p-q$. In the following we suppose $p > 2$, $q > 2$.

Considering the infinitesimal product conformal and infinitesimal product projective transformations on a locally decomposable Riemannian space, S.Tachibana [2] obtained the product conformal curvature tensor.

$$\begin{aligned}
C(X,Y,Z) &= K(X,Y,Z) \\
&+ a[K(X,Y)Z - g(X,Z)K(Y) + K(X,FY)FZ - g(X,FZ)K(FY) \\
&- K(X,Z)Y + g(X,Y)K(Z) - K(X,FZ)FY + g(X,FY)K(FZ)] \\
&- b[K(X,FY)Z - g(X,FZ)K(Y) + K(X,Y)FZ - g(X,Z)K(FY) \\
&- K(X,FZ)Y + g(X,FY)K(Z) - K(X,Z)FY + g(X,Y)K(FZ)] \\
&+ [K(a\alpha_1 - b\beta_1) + K^*(a\beta_1 - b\alpha_1)][g(X,Z)Y - g(X,Y)Z \\
&+ g(X,FZ)FY - g(X,FY)FZ] + [K(a\beta_1 - b\alpha_1) + \\
&+ K^*(a\alpha_1 - b\beta_1)][g(X,FZ)Y - g(X,FY)Z + g(X,Z)FY - g(X,Y)FZ]
\end{aligned} \tag{1}$$

and product projective curvature tensor

$$\begin{aligned}
P(X,Y,Z) &= K(X,Y,Z) \\
&+ \alpha_1[K(X,Z)Y - K(X,Y)Z + K(X,FZ)FY - K(X,FY)FZ] \\
&+ \beta_1[K(X,FZ)Y - K(X,FY)Z + K(X,Z)FY - K(X,Y)FZ],
\end{aligned} \tag{2}$$

where $K(X,Y,Z)$, $K(X,Y)$ and K are the curvature tensor, the Ricci tensor and scalar curvature of the Riemannian space, $K^* = K^a_b F^b_a$ and

$$a = \frac{n-4}{\phi^2 - (n-4)^2}, \qquad b = \frac{\phi}{\phi^2 - (n-4)^2} \tag{3}$$

$$(4) \qquad \alpha_1 = \frac{n-2}{(n-2)^2-\phi^2}, \qquad \beta_1 = -\frac{\phi}{(n-2)^2-\phi^2}.$$

In [3] the product concircular curvature tensor is obtained:

$$S(X,Y,Z) = K(X,Y,Z)$$
$$(5) \qquad - (\alpha K+\beta K^*)\left[g(X,Z)Y - g(X,Y)Z + g(X,FZ) - g(X,FY)FZ\right]$$
$$- (\beta K+\alpha K^*)\left[g(X,FZ)Y - g(X,FY)Z + g(X,Z)FY - g(X,Y)FZ\right],$$

where

$$(6) \qquad \alpha = \frac{n(2-n)-\phi^2}{(n^2-\phi^2)\left[(2-n)^2-\phi^2\right]}, \qquad \beta = -\frac{2\phi(1-n)}{(n^2-\phi^2)\left[(2-n)^2-\phi^2\right]}.$$

In [4] the product conharmonic curvature tensor is defined as follows:

$$W(X,Y,Z) = K(X,Y,Z)$$
$$+ a\left[K(X,Y)Z - g(X,Z)K(Y) + K(X,FY)FZ - g(X,FZ)K(FY)\right.$$
$$(7) \qquad \left. - K(X,Z)Y + g(X,Y)K(Z) - K(X,FZ)FY + g(X,FY)K(FZ)\right]$$
$$- b\left[K(X,FY)Z - g(X,FZ)K(Y) + K(X,Y)FZ - g(X,Z)K(FY)\right.$$
$$\left. - K(X,FZ)Y + g(X,FY)K(Z) - K(X,Z)FY + g(X,Y)K(FZ)\right].$$

The purpose of the present note is to find some relations connecting the tensors $C(X,Y,Z)$, $P(X,Y,Z)$, $S(X,Y,Z)$ and $W(X,Y,Z)$.

2. First we note that the contraction of (2) with respect to Z gives the zero tensor.

We transvect (2) by ${}^{-1}g(X,Y)$, where ${}^{-1}g$ is the conjugate tensor of g, and denote the obtained tensor by $P(Z)$. Then we have

$$P(Z) = (1+2\alpha_1)K(Z) + 2\beta_1 K(FZ) - (\alpha_1 K+\beta_1 K^*)Z - (\alpha_1 K^*+\beta_1 K)FZ$$

and

$$(8) \qquad P(X,Z) = (1+2\alpha_1)K(X,Z) + 2\beta_1 K(X,FZ) - (\alpha_1 K + \beta_1 K^*)g(X,Z) - (\alpha_1 K^*+\beta_1 K)g(X,FZ)$$

where

$$P(X,Z) = g(X,P(Z)).$$

Now, we calculate the tensor

$$P(X,Y,Z) + A[P(X,Z)Y - P(X,Y)Z + P(X,FZ)FY - P(X,FY)FZ] + B[P(X,FZ)Y - P(X,FY)Z + P(X,Z)FY - P(X,Y)FZ],$$

where A and B are some constants. Taking into account (2) and (8), we find that this tensor can be expressed in the form

$$\begin{aligned} &K(X,Y,Z) + \\ &+ [\alpha_1 + A(1+2\alpha_1) + 2B\beta_1][K(X,Z)Y-K(X,Y)Z+K(X,FZ)FY- \\ &- K(X,FY)FZ]+[\beta_1+2A\beta_1+B(1+2\alpha_1)][K(X,FZ)Y-K(X,FY)Z + \\ &+ K(X,Z)FY-K(X,Y)FZ] - [(A\alpha_1+B\beta_1)K+K(A\beta_1+B\alpha_1)K^*] \\ &\cdot [g(X,Z)Y-g(X,Y)Z+g(X,FZ)FY-g(X,FY)FZ] \\ &-[(A\beta_1+B\alpha_1)K+(A\alpha_1+B\beta_1)K^*][g(X,FZ)Y- \\ &- g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ] . \end{aligned}$$

If we determine the numbers A and B such that

$$\alpha_1 + A(1+2\alpha_1) + 2B\beta_1 = 0,$$

$$\beta_1 + 2A\beta_1 + B(1+2\alpha_1) = 0,$$

we get

$$(9) \qquad A = \frac{2(\beta_1^2 - \alpha_1^2)-\alpha_1}{(1+2\alpha_1)^2-4\beta_1^2}, \qquad B = -\frac{\beta_1}{(1+2\alpha_1^2)^2-4\beta_1^2} .$$

By straight forward calculations we find

$$(1+2\alpha_1)^2 - 4\beta_1^2 = \frac{n^2-\phi^2}{(n-2)^2-\phi^2} .$$

Since $n \neq \phi$, $n^2-\phi^2 \neq 0$. Since $p > 2$ and $q > 2$, $(n-2)^2-\phi^2 \neq 0$.

Also, by direct calculation, we obtain

$$A\alpha_1 + B\beta_1 = \alpha , \qquad A\beta_1 + B\alpha_1 = \beta .$$

Therefore, we find

$$\begin{aligned} &P(X,Y,Z) + A\left[P(X,Z)Y-P(X,Y)Z+P(X,FZ)FY-P(X,FY)FZ\right] \\ (10) \qquad &+ B\left[P(X,FZ)Y-P(X,FY)Z+P(X,Z)FY-P(X,Y)FZ\right] = S(X,Y,Z) \end{aligned}$$

where the numbers A and B have the values (9).

3. We contract (5) with respect to Z and denote by $S(X,Y)$ the obtained tensor. Then we have

$$\begin{aligned} S(X,Y) = K(X,Y) &+ \{[\alpha(n-2)+\phi\beta]K+[\beta(n-2)+\phi\alpha]K^*\}g(X,Y) \\ &+\{[\phi\alpha+(n-2)\beta]K+[\phi\beta+\alpha(n-2)]K^*\}g(X,FY) \ . \end{aligned}$$

Taking into account (6), we have

$$\alpha(n+2)+\phi\beta = -\frac{n}{n^2-\phi^2}\ , \qquad \alpha\phi+\beta(n-2) = \frac{\phi}{n^2-\phi^2}\ .$$

Therefore

$$\begin{aligned} (11) \qquad S(X,Y) = K(X,Y) &+ \frac{1}{n^2-\phi^2}(-nK+\phi K^*)g(X,Y)+ \\ &+\frac{1}{n^2-\phi^2}(\phi K-nK^*)g(X,FY)\ . \end{aligned}$$

Taking into account (5) and (11), we get

$$\begin{aligned} S(X,Y,Z) &+ a\left[S(X,Y)Z-g(X,Z)S(Y)+S(X,FY)FZ-g(X,FZ)S(FY)\right. \\ &\left.- S(X,Z)Y+g(X,Y)S(Z)-S(X,FZ)FY+g(X,FY)S(FZ)\right] \\ &- b\left[S(X,FY)Z-g(X,FZ)S(Y)+S(X,Y)FZ-g(X,Z)S(FY)\right. \\ &\left.- S(X,FZ)Y+g(X,FY)S(Z)-S(X,Z)FY+g(X,Y)S(FZ)\right] \\ = K(X,Y,Z) &+ a\left[K(X,Y)Z-g(X,Z)K(Y)+K(X,FY)FZ-g(X,FZ)K(FY)\right. \\ &\left.- K(X,Z)Y+g(X,Y)K(Z)-K(X,FZ)FY+g(X,FY)K(FZ)\right] \\ &- b\left[K(X,FY)Z-g(X,FZ)K(Y)+K(X,Y)FZ-g(X,Z)K(FY)\right. \\ &\left.- K(X,FZ)Y+g(X,FY)K(Z)-K(X,Z)FY+g(X,Y)K(FZ)\right] \\ &+ \left[(-\alpha+\frac{2(an+\phi b)}{n^2-\phi^2})K - (\beta+\frac{2(a\phi+bn)}{n^2-\phi^2})K^*\right]\left[g(X,Z)Y-g(X,Y)Z+\right. \\ &\left.+ g(X,FZ)FY-g(X,FY)FZ\right] + \left[-(\beta+\frac{2(a\phi+bn)}{n^2-\phi^2})K+\right. \\ &\left.+ (-\alpha+\frac{2(an+b\phi)}{n^2-\phi^2})K^*\right]\left[g(X,FZ)Y-g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ\right]. \end{aligned}$$

But

$$-\alpha + \frac{2(an+\phi b)}{n^2-\phi^2} = a\alpha_1 - b\beta_1$$

and

$$-(\beta + \frac{2(a\phi + bn)}{n^2-\phi^2}) = a\beta_1 - b\alpha_1$$

because of (3) and (6). Therefore it follows that:

$$\begin{aligned}
&S(X,Y,Z) + a[S(X,Y)Z-g(X,Z)S(Y)+S(X,FY)FZ-g(X,FZ)S(FY)\\
&\quad - S(X,Z)Y+g(X,Y)S(Z)-S(X,FZ)FY+g(X,FY)S(FZ)]\\
(12)\quad &\quad - b[S(X,FY)Z-g(X,FZ)S(Y)+S(X,Y)FZ-g(X,Z)S(FY)\\
&\quad - S(X,FZ)Y+g(X,FY)S(Z)-S(X,Z)FY+g(X,Y)S(FZ)]\\
&\quad = C(X,Y,Z).
\end{aligned}$$

4. Taking into account (5) and (11), we find:

$$\begin{aligned}
&S(X,Y,Z)+\alpha_1[S(X,Z)Y-S(X,Y)Z+S(X,FZ)FY-S(X,FY)FZ]\\
&\quad +\beta_1[S(X,FZ)Y-S(X,FY)Z+S(X,Z)FY-S(X,Y)FZ]\\
&= K(X,Y,Z)+\alpha_1[K(X,Z)Y-K(X,Y)Z+K(X,FZ)FY-K(X,FY)FZ]\\
&\quad +\beta_1[K(X,FZ)Y-K(X,FY)Z+K(X,Z)FY-K(X,Y)FZ]\\
&\quad +[(-\alpha+\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2})K + (-\beta + \frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2})K^*][g(X,Z)Y\\
&\quad -g(X,Y)Z+g(X,Z)FY-g(X,FY)FZ] + [(-\beta+\frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2})K +\\
&\quad + (-\alpha+\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2})K^*][g(X,FZ)Y-g(X,FY)Z +\\
&\quad + g(X,Z)FY-g(X,Y)FZ] .
\end{aligned}$$

But

$$\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2} = \alpha \quad \text{and} \quad \frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2} = \beta$$

because of (6). Therefore it follows that:

$$\begin{aligned}
(13)\quad &S(X,Y,Z) + \alpha_1[S(X,Z)Y-S(X,Y)Z+S(X,FZ)FY-S(X,FY)FZ]\\
&\quad + \beta_1[S(X,FZ)Y-S(X,FY)Z+S(X,Z)FY-S(X,Y)FZ]\\
&\quad = P(X,Y,Z) .
\end{aligned}$$

5. Contracting (7) with respect to Z, we obtain

$$W(X,Y) = [1+a(n-4)-b\phi]K(X,Y)+[a\phi-b(n-4)]K(X,FY)$$
$$+ (aK-bK^*)g(X,Y)+(aK^*-bK)g(X,FY).$$

It is easy to see that

$$1 + a(n-4) - b\phi = 0 \quad \text{and} \quad a\phi - b(n-4) = 0.$$

Therefore

$$W(X,Y) = (aK-bK^*)g(X,Y) + (aK^*-bK)g(X,FY).$$

Taking this into account, as well as (7), we have

$$W(X,Y,Z)+\alpha_1 [W(X,Z)Y-W(X,Y)Z+W(X,FZ)FY-W(X,FY)FZ]$$
$$+\beta_1 [W(X,FZ)Y-W(X,FY)Z+W(X,Z)FY-W(X,Y)FZ]=W(X,Y,Z)$$
$$+[(a\alpha_1-b\beta_1)K+(-b\alpha_1+a\beta_1)K^*] [g(X,Z)Y-g(X,Y)Z+g(X,FZ)FY-$$
$$- g(X,FY)FZ]+[(-b\alpha_1+a\beta_1)K +(a\alpha_1-b\beta_1)K^*] [g(X,FZ)Y-$$
$$- g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ]$$

i.e.

(14)
$$W(X,Y,Z)+\alpha_1 [W(X,Z)Y-W(X,Y)Z+W(X,FZ)FY-W(X,FY)FZ]$$
$$+\beta_1 [W(X,FZ)Y-W(X,FY)Z+W(X,Z)FY-W(X,Y)FZ]$$
$$= C(X,Y,Z).$$

6. Taking into account (8) and (11), we can easily see that

(15)
$$P(X,Y) = (1+2\alpha_1)S(X,Y) + 2\beta_1 S(X,FY)$$

and

(16)
$$S(X,Y) = \frac{1+2\alpha_1}{(1+2\alpha_1)^2-4\beta_1^2} P(X,Y) - \frac{2\beta_1}{(1+2\alpha_1)^2-4\beta_1^2} P(X,FY).$$

7. Relations (10),(12),(13),(14),(15) and (16) are the required relations connecting the tensors $C(X,Y,Z)$, $P(X,Y,Z)$, $S(X,Y,Z)$ and $W(X,Y,Z)$.

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REZIME

JEDNA PRIMEDBA O PRODUKT-TENZORIMA KRIVINE

Dokazane su relacije (10), (12), (13), (14) i (15) koje povezuju sledeće tenzore: tenzor produkt-konforme krivine (1), tenzor produkt-projektivne krivine (2), tenzor produkt-koncirkularne krivine (5) i tenzor produkt-konharmonijske krivine (7).
(7).

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AUTOPARALLEL CURVES OF RIEMANN-OTSUKI SPACES

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ABSTRACT

In this paper we study a Riemann-Otsuki space $R\text{-}O_n$ and one of its m-dimensional ($m < n$) subspaces, which is also a Riemann-Otsuki space, (see [1] and [2]). We denote that subspace by $R\text{-}O_m$. Our aim is to determine the conditions by which the autoparallel curves of $R\text{-}O_m$ are the autoparallel curves of $R\text{-}O_n$, too.

In [4] the author considers the autoparallel curves of Weyl-Otsuki spaces. Using the fact that the coefficients of the connection of the covariant and contravariant parts of Otsuki´s spaces are different, he gives the autoparallel curves of the covariant and contravariant kind respectively. Following this way, we shall study autoparallel curves of the covariant kind in paragraph 1, and in paragraph 2 we shall consider autoparallel curves of the contravariant kind. In paragraphs 3 and 4, we shall observe the above two kinds of autoparallel curves, especially if the subspace has an intrinsic or induced connection respectively.

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PRELIMINARIES

The theory of Weyl-Otsuki spaces was laid down by A. Moór in [3]. We get $R\text{-}O_n$ spaces from $W\text{-}O_n$ spaces if we suppose that in the relation $\nabla_k g_{ij} = \gamma_k g_{ij}$ it holds that $\gamma_k = 0$. Namely, the $R\text{-}O_n$ space is an n-dimensional differentiable manifold with Riemannian metric tensor g_{ij}, $\det(g_{ij}) \neq 0$ and Otsuki's connection. The basic elements of the $R\text{-}O_n$ space are g_{ij} and the tensor P^i_j, $\det(P^i_j) \neq 0$. As in [3] and [5] the invariant differential in the spaces of the Otsuki kind with the coordinates x^i is defined by

$$(0.1) \qquad DT^i_j := P^i_a P^b_j \bar{D} T^a_b$$

where

$$(0.2) \qquad \bar{D} T^a_b := (\partial_k T^a_b + {}'\Gamma^a_{sk} T^s_b - {}''\Gamma^s_{bk} T^a_s) dx^k .$$

Tensor P^i_j and the coefficients of connections ${}'\Gamma^i_{jk}$ and ${}''\Gamma^i_{jk}$ satisfy Otsuki's relation

$$(0.3) \qquad \partial_k P^i_j - {}'\Gamma^t_{jk} P^i_t + {}''\Gamma^i_{tk} P^t_j = 0 .$$

We suppose that the tensor P^i_j has an inverse Q^i_j and the relations

$$(0.4) \quad \text{a)} \quad P^i_j Q^j_s = \delta^i_s , \qquad \text{b)} \quad P^i_j g_{ia} = P^i_a g_{ij}$$

hold.

We define the subspace in $R\text{-}O_n$ by the relation

$$(0.5) \qquad x^i = x^i(u^1, \dots, u^m) \quad (m < n) .$$

By our suppositon $\operatorname{rank}(\partial x^i / \partial u^\alpha) = m$ 1/, and we use the notation

$$(0.6) \qquad B^i_\alpha := \frac{\partial x^i}{\partial u^\alpha} .$$

The metric tensor of the subspace $R\text{-}O_m$ is defined as usually by

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1/. *In this article Latin indices run from 1 to n and Greek indices $\alpha, \beta, \dots, \lambda$ run from 1 to m, but $\mu, \nu, \dots, \omega$ run from (m+1) to n.*

(0.7) $$G_{\alpha\beta} := g_{ij} B^i_\alpha B^j_\beta \, .$$

The basic tensor P^α_β of the subspace R-O_m is defined by the projection of P^i_j on the subspace and

(0.8) $$P^\alpha_\beta := P^i_j B^\alpha_i B^j_\beta$$

where

(0.9) $$B^\alpha_i := g_{ij} G^{\alpha\beta} B^j_\beta \, .$$

We define the inverse tensor of the tensor P^α_β by $\overset{*}{Q}{}^\alpha_\beta$, i.e.

(0.10) $$P^\alpha_\beta \overset{*}{Q}{}^\beta_\gamma = \delta^\alpha_\gamma \, .$$

As in the embedding space, in the subspace we define the invariant differential $\overset{*}{D}$ of the tensor T^α_β defined over the subspace by

(0.11) $$\overset{*}{D}T^\alpha_\beta := P^\alpha_\gamma P^\lambda_\beta \overset{*}{\bar{D}} T^\gamma_\lambda$$

where

(0.12) $$\overset{*}{\bar{D}} T^\gamma_\lambda := (\partial_\chi T^\gamma_\lambda + {}'\Gamma^\gamma_{\varepsilon\chi} T^\varepsilon_\lambda - {}''\overset{*}{\Gamma}{}^\varepsilon_{\lambda\chi} T^\gamma_\varepsilon) du^\chi \, .$$

We suppose that the tensor $G_{\alpha\beta}$ is a metric tensor of the Riemannian kind i.e. $\det(G_{\alpha\beta}) \neq 0$ and $\overset{*}{D}G_{\alpha\beta} = 0$. From this condition, using (0.12), we shall determine ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and by using Otsuki´s relation analogous to (0.3) for P^α_β, ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ we get ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ (see [1]). We can determine the coefficients of the connections of the subspace in other ways, too. This will be seen in paragraph 4.

Using the tangent vectors B^i_α we can determine the vectors N^μ_i orthogonal to the subspace R-O_m by the equations $B^i_\alpha N^\mu_i = 0$ and we get

(0.13) $$\delta^i_j = B^i_\alpha B^\alpha_j + N^i_\mu N^\mu_j \, .$$

It is known that if $m \neq n-1$ the vectors N_i^{μ} are not uniquely determined.

1. COVARIANT TYPE OF AUTOPARALLEL CURVES

We shall now consider the subspace $R\text{-}O_m$ defined by relation (0.5). The curve $C: u^{\alpha}(S)$ is an autoparallel curve of the subspace if the tangent vector du^{α}/ds /2 is a parallel displaced along C. Applying (0.11) and (0.12) in $\frac{\overset{*}{D}}{ds}(du^{\alpha}/ds) = 0$, contracting by $\overset{*}{Q}{}^{\alpha}_{\beta}$ and using (0.10) we get an equation of the autoparallel curve of a contravariant type in the form

$$\frac{d^2u^{\alpha}}{ds^2} + \overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0 \; . \tag{1.1}$$

We ask under which conditions will the autoparallel curve of the observed type of the subspace be, at the same time, the autoparallel curve of this type in an embedding space, too.

Let

$$C: x^i = x^i(u^{\alpha}(s)) \tag{1.2}$$

be the autoparallel curve of the subspace $R\text{-}O_m$. Using the differential quotient of (1.2), applying (0.6) we get

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^{\alpha}}\frac{du^{\alpha}}{ds} = B^i_{\alpha}\frac{du^{\alpha}}{ds}$$

and

$$\frac{d^2x^i}{ds^2} = \frac{\partial B^i_{\alpha}}{\partial u^{\beta}}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} + B^i_{\alpha}\frac{d^2u^{\alpha}}{ds^2} \; . \tag{1.3}$$

Since, according to our supposition, C is the autoparallel curve of the covariant type, eliminating d^2u^{α}/ds^2 with (1.1) from (1.3) we get

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2/ s always denotes the arc length as parameter.

$$\frac{d^2x^i}{ds^2} = \left(\frac{\partial B^i_\beta}{\partial u^\gamma} - B^i_\alpha \,{}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}(u(s))\right) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} \quad .$$

Hence if C is the autoparallel curve in the space $R\text{-}O_n$, it must be

$$(1.4) \qquad \left(\frac{\partial B^i_\beta}{\partial u^\gamma} + {}'\Gamma^i_{sk} B^s_\beta B^k_\gamma \right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = B^i_\alpha \,{}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} \quad .$$

Now we can formulate.

THEOREM 1. *Relation (1.4) is a necessary and sufficient condition for curve C to be the autoparallel curve of the contravariant type on the subspace $R\text{-}O_m$ and in the embedding space $R\text{-}O_n$ too.*

P r o o f. It follows from the above condition, that the condition is sufficient. Now we shall prove that it is neces - sary, too. From (1.3) and the supposition that curve C is autoparallel in $R\text{-}O_n$, it follows that

$$-\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \frac{\partial B^i_\alpha}{\partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + B^i_\alpha \frac{d^2u^\alpha}{ds^2}$$

or

$$B^i_\alpha \frac{d^2u^\alpha}{ds^2} = -\left({}'\Gamma^i_{jk} B^j_\alpha B^k_\beta + \frac{\partial B^i_\alpha}{\partial u^\beta} \right) \frac{du^\alpha}{ds}\frac{du^\beta}{ds} \quad .$$

Substituting (1.4) and contracting by B^δ_i, we get (1.1) and curve C is autoparallel on $R\text{-}O_m$. It is obvious that (1.1), (1.3) and $d^2x^i/ds^2 + {}'\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$ do not hold at the same time if (1.4) does not hold.

After general theorem 1, we shall investigate same special cases.

THEOREM 2. *If (1.4) holds for curve C of the subspace $R\text{-}O_m$ and the vector $\xi^i = B^i_\alpha \xi^\alpha$ is a vector of the subspace defined along curve C in its direction, then along C it holds*

$$(1.5) \qquad \frac{\overset{*}{D}\xi^\alpha}{ds} = P^\alpha_\lambda B^\lambda_i Q^i_a \frac{D\,\xi^a}{ds} \quad .$$

P r o o f. From (1.4) it follows that the considered curve is autoparallel in the subspace and in the embedding space, too. Using definitions (0.2) and (0.6), the basic invariant differential quotient of ξ^i in R-O_n is

$$\frac{\bar{D}\xi^i}{ds} = \frac{d(B^i_\beta \xi^\beta)}{ds} + \acute{\Gamma}^{\;i}_{sk} B^s_\alpha \xi^\alpha B^k_\beta \frac{du^\beta}{ds} .$$

Multiplying by $g_{ij}B^j_\alpha$ we get

$$(1.6) \quad g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = g_{ij}B^j_\alpha \Big[\frac{d\xi^\beta}{ds} B^i_\beta + \Big(\frac{\partial B^i_\beta}{\partial u^\gamma} + \acute{\Gamma}^i_{sk} B^s_\beta B^k_\gamma\Big)\xi^\beta \frac{du^\gamma}{ds} \Big] .$$

According to the stipulation of the theorem, vector ξ^α satisfies

$$(1.7) \qquad \xi^\alpha = \xi \frac{du^\alpha}{ds}$$

and (1.4) holds. Substituting (1.7) and (1.4) we get

$$(1.8) \qquad g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = g_{ij}B^j_\alpha B^i_\beta \Big[\frac{d\xi^\beta}{ds} + \acute{\overset{*}{\Gamma}}{}^\beta_{\gamma\chi} \xi^\gamma \frac{du^\chi}{ds} \Big] .$$

Using definitions (0.7) and (0.12) we get

$$g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = G_{\alpha\beta} \frac{\bar{\overset{*}{D}}\xi^\beta}{ds} .$$

Expressing the basic covariant differential quotient $\bar{D}/ds$ by the covariant differential quotient D/ds and contracting by $G^{\alpha\delta}$ we get

$$g_{ij}B^j_\alpha G^{\alpha\delta} Q^i_t \frac{D\xi^t}{ds} = \overset{*}{Q}{}^\beta_\gamma \frac{\overset{*}{D}\xi^\gamma}{ds} .$$

Using definition (0.9) and contracting by P^α_β according to (0.10) we finally get (1.5).

Relation (1.5) means that the covariant differential of the contravariant vector in our subspace does not depend only on the projection of the covariant differential of the space R-O_n, but also on the tensor P^α_β of subspace R-O_m and on tensor Q^i_a which is the inverse of tensor P^i_j of the R-O_n space.

Instead condition (1.4) it is possible to take a stronger condition and formulate the following

THEOREM 3. *If in the subspace* $R\text{-}O_m$ *along* C *we suppose that condition*

$$B^i_\alpha \,{}^{*}\!\Gamma^\alpha_{\beta\gamma}(u)\,\frac{du^\gamma}{ds} = \left[\frac{\partial B^i_\beta}{\partial u^\gamma} + {}'\Gamma^i_{jk}(x)B^j_\beta B^k_\gamma\right]\frac{du^\gamma}{ds} \tag{1.9}$$

holds, then (1.5) holds for the optional vector ξ^α *defined along* C.

P r o o f. Condition (1.9) is stronger than condition (1.4) and it follows from this that curve C is autoparallel in $R\text{-}O_m$ and $R\text{-}O_n$. A calculation analogous to the above gives (1.6). Using (1.9) we get (1.8). It is not difficult to see that this is identical to (1.5).

2. COVARIANT TYPE OF AUTOPARALLEL CURVES

Curves satisfying relation

$$\frac{D}{ds}\left(g_{ij}(x)\,\frac{dx^j}{ds}\right) = 0 \tag{2.1}$$

will be called autoparallel curves of a covariant type. In Riemannian spaces this equation is equivalent to relation $\frac{D}{ds}\left(\frac{dx^j}{ds}\right) = 0$ because $Dg_{ij} = 0$ and the Leibniz formula holds. Applying definition (0.1) and using the contraction by Q^i_r according to (0.4), from (2.1) we get

$$\frac{dg_{rj}}{ds}\frac{dx^j}{ds} + g_{rj}\frac{d^2x^j}{ds} - {}''\Gamma_{rjk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Multiplying by g^{ir} and using the proposition that in $R\text{-}O_n$ spaces $\bar{D}g_{ij} = 0$, we get

$$\frac{d^2x^i}{ds^2} + {}''\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{2.2}$$

(see [4] (3.2) and (3.2a) with $\gamma_k = 0$). This is the equation of the autoparallel curve of the covariant type in R-O_n. The equation of the autoparallel curve of the covariant type in subspace R-O_m is

$$(2.3)\qquad \frac{d^2u^\alpha}{ds} + "\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0,$$

because $\overset{*}{\bar{D}}G_{\alpha\beta} = 0$ and s is the arc lenght as parameter. Substituting d^2u^α/ds^2 from (2.3) in (1.3) we get

$$\frac{d^2x^i}{ds^2} = \left(\frac{\partial B^i_\alpha}{\partial u^\beta} - B^i_\gamma "\overset{*}{\Gamma}{}^\gamma_{\alpha\beta}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}.$$

From this follows the condition of the covariant case, which is analogous to (1.4). This is

$$(2.4)\qquad \left(\frac{\partial B^i_\alpha}{\partial u^\beta} + "\Gamma^i_{jk}B^j_\alpha B^k_\beta\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = B^i_\gamma "\overset{*}{\Gamma}{}^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}.$$

It is not difficult to see that the following holds.

THEOREM 4. *Condition (2.4) is sufficient and necessary so that the autoparallel curve of covariant type of the subspace should be the autoparallel curve of the embedding space, too.*

Now we shall study whether theorems analogous to theorems 2 and 3 of the first paragraph hold in this case, too. Applying definition (0.2) on vector ξ_i, which satisfies $\xi_i = B^\alpha_i\xi_\alpha$, and multiplying $\bar{D}\xi_i/ds$ by $g^{ij}B^\alpha_j$ we get

$$(2.5)\qquad g^{ij}B^\alpha_j\frac{\bar{D}\xi_i}{ds} = g^{ij}B^\alpha_j\left[\frac{d\xi_\beta}{ds}B^\beta_i + \left(\frac{\partial B^\beta_i}{\partial u^\chi} - "\Gamma^s_{ik}B^\beta_s B^k_\chi\right)\xi_\beta\frac{du^\chi}{ds}\right].$$

At first, using the definition of B^β_i we galculate

$$g^{ij}B^\alpha_j\frac{\partial B^\beta_i}{\partial u^\chi} = g^{ij}B^\alpha_j\left[\frac{\partial g_{ir}}{\partial x^k}B^k_\chi B^r_\gamma G^{\gamma\beta} + \frac{\partial B^r_\gamma}{\partial u^\chi}g_{ir}G^{\gamma\beta} + g_{ir}B^r_\gamma\frac{\partial G^{\gamma\beta}}{\partial u^\chi}\right].$$

Using that $\bar{D}g_{ir} = 0$, the definition of B^β_s and $B^\alpha_r B^r_\gamma = \delta^\alpha_\gamma$ we get

$$g^{ij}B^{\alpha}_{j}\left(\frac{\partial B^{\beta}_{i}}{\partial u^{\chi}} - {}''\Gamma^{s}_{ik}B^{k}_{\chi}B^{\beta}_{s}\right) = \left({}''\Gamma^{j}_{rk}B^{r}_{\gamma}B^{k} + \frac{\partial B^{j}_{\gamma}}{\partial u^{\chi}}\right) B^{\alpha}_{j}G^{\gamma\beta} + \frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} \quad .$$

Substituting it in (2.5) using $G^{\alpha\beta}$, which we get from (0.7), we have

$$g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{d\xi_{\beta}}{ds} + \left({}''\Gamma^{j}_{rk}B^{r}_{\gamma}B^{k}_{\chi} + \frac{\partial B^{j}_{\gamma}}{\partial u^{\chi}}\right) B^{\alpha}_{j}G^{\gamma\beta}\xi_{\beta}\frac{du^{\chi}}{ds} + \frac{\partial G^{\alpha\beta}}{\partial u^{\chi}}\xi_{\beta}\frac{du^{\chi}}{ds} \ .$$

Finally we add and subtract $G^{\alpha\beta}\,{}''\overset{*}{\Gamma}{}^{\lambda}_{\beta\chi}\xi_{\lambda}\dfrac{du^{\chi}}{ds}$ and we get

$$(2.6)\quad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\bar{D}\xi_{\beta}}{ds} + \Big[\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + \left({}''\Gamma^{i}_{jk}B^{j}_{\gamma}B^{k}_{\chi} + \frac{\partial B^{i}_{\gamma}}{\partial u^{\chi}}\right)B^{\alpha}_{i}G^{\gamma\beta}\Big]\xi_{\beta}\frac{du^{\chi}}{ds} \ .$$

Using the property that ξ_{β} is a vector tangential to the observed curve C and $G^{\gamma\beta}\xi_{\beta} = \xi^{\gamma} = \xi\dfrac{du^{\gamma}}{ds}$ and using the proposition that condition (2.4) holds, from the above equation we get

$$(2.7)\quad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\overset{*}{\bar{D}}\xi_{\beta}}{ds} + \left(\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + {}''\overset{*}{\Gamma}{}^{\alpha}_{\gamma\chi}G^{\gamma\beta}\right)\xi_{\beta}\frac{du^{\chi}}{ds} \ .$$

It is known, that in Otsuki´s space it is possible to define the covariant and basic covariant differential with respect only to one of the coefficients of connections. We denote these differentials by ´D and "D or ´$\bar{D}$ and "$\bar{D}$ respectively. Hence we see that

$$\left(\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + {}''\overset{*}{\Gamma}{}^{\alpha}_{\gamma\chi}G^{\gamma\beta}\right)\frac{du^{\chi}}{ds} = \frac{{}''\overset{*}{\bar{D}}G^{\alpha\beta}}{ds} \ .$$

Since we know that in the observed space $\overset{*}{\bar{D}}G_{\alpha\beta} = {}''\overset{*}{\bar{D}}G_{\alpha\beta} = 0$ and $G_{\alpha\beta}G^{\beta\gamma} = \delta^{\gamma}_{\alpha}$, one can see that ${}''\overset{*}{\bar{D}}G^{\alpha\beta}/ds = 0$. Now from (2.7) it follows

$$(2.8)\qquad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\overset{*}{\bar{D}}\xi_{\beta}}{ds} \ .$$

Substituting B^{α}_{j} from (0.9) and contracting by $G_{\alpha\gamma}$, we get

$B^i_\gamma \frac{\bar{D}\xi_i}{ds} = \frac{\overset{*}{\bar{D}}\xi_\gamma}{ds}$. Using (0.1) and (0.4) or (0.11) and (0.10) respectively, the basic invariant differentials $\bar{D}$, $\overset{*}{\bar{D}}$ can be expressed by the differentials D, $\overset{*}{D}$ respectively and so $\frac{\bar{D}\xi_i}{ds} = Q^r_i \frac{D\xi_r}{ds}$ and $\frac{\overset{*}{\bar{D}}\xi_\gamma}{ds} = \overset{*}{Q}{}^\alpha_\gamma \frac{\overset{*}{D}\xi_\alpha}{ds}$. Finally from (2.8) we get

$$\frac{\overset{*}{D}\xi_\alpha}{ds} = P^\gamma_\alpha B^i_\gamma Q^r_i \frac{D\xi_r}{ds} \tag{2.9}$$

and it is possible to formulate

THEOREM 4. *If in subspace* $R\text{-}O_m$, *(2.4) holds, vector* ξ_α *is a vector defined along curve* C *in its direction and* $\xi_r = B^\alpha_r \xi_\alpha$, *then along* C *(2.9) holds.*

The above theorem can also possible be formulated along C for all vectors of subspace $R\text{-}O_m$, but with a condition stronger than (2.4). This condition is

$$B^i_\alpha \, {}''\overset{*}{\Gamma}{}^\alpha_{\beta\chi} \frac{du^\chi}{ds} = \left[\frac{\partial B^i_\beta}{\partial u^\chi} + {}''\Gamma^i_{jk} B^j_\beta B^k_\chi \right] \frac{du^\chi}{ds} \,. \tag{2.10}$$

Substituting the right side of (2.10) in (2.6) and using the fact that in our space $\frac{{}''\bar{D}G^{\alpha\beta}}{ds} = 0$, we get (2.8). So the following holds.

THEOREM 5. *From (2.10) it follows that for vector* ξ_α *of the subspace, components of which in the embedding space* $R\text{-}O_n$ *are* $\xi_i = B^\alpha_i \xi_\alpha$, *along the autoparallel curves of the subspace, (2.9) holds.*

3. SPECIAL CASES WITH AN INTRINSIC CONNECTION OF THE SUBSPACE

In article [1] the author gives the formulae of the coefficients of connections ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$. In these formulae the coefficients of connections ${}'\Gamma^i_{jk}$ and ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ or ${}''\Gamma^i_{jk}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$

respectively are connected in a special way. Indeed, the coefficients of connection ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ in this case are the coefficients of intrinsic connection of subspace $R\text{-}O_m$. Substituting ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ in conditions (1.4) and (2.4) respectively, we get conditions equivalent to them, which we denote by $(1.\overset{*}{}4)$ and $(2.\overset{*}{}4)$ respectively.

At first we observe autoparallel curves of the contravatiant type. From [1] (24) we have that

$$ {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = \overset{*}{Q}{}^{\beta}_{\alpha}B^{\alpha}_{i}\left[B^{j}_{\delta}B^{k}_{\gamma}P^{i}_{a}\,{}'\Gamma^{a}_{jk} - P^{a}_{j}B^{j}_{\delta}N^{\mu}_{a}N^{b}_{\mu}\,{}''\Gamma^{i}_{bc}B^{c}_{\gamma} - \right.$$
$$\left. - P^{a}_{j}B^{j}_{\delta}N^{\mu}_{a}(\partial_{\gamma}N^{i}_{\mu}) + P^{i}_{j}B^{j}_{\delta\gamma}\right] .$$

Using transformation $u^{\alpha} \to u^{\alpha'}$ of the coordinates we see that ${}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}$ given in the above form change themselves in the following way

$$ {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = {}'\overset{*}{\Gamma}{}^{\beta'}_{\delta'\gamma'}\cdot\ \frac{\partial u^{\beta}}{\partial u^{\beta'}}\frac{\partial u^{\delta'}}{\partial u^{\delta}}\frac{\partial u^{\gamma'}}{\partial u^{\gamma}} + \frac{\partial u^{\beta}}{\partial u^{\beta'}}\frac{\partial^{2}u^{\delta'}}{\partial u^{\gamma}\partial u^{\delta}}P^{i}_{b}B^{\alpha'}_{i}\overset{*}{Q}{}^{\beta'}_{\alpha'}B^{b}_{\delta'} \ . \tag{3.1}$$

This is the transformation form of the coefficients of connection iff

$$P^{i}_{b}B^{\alpha'}_{i}\overset{*}{Q}{}^{\beta'}_{\alpha'}B^{b}_{\delta'} = \delta^{\beta'}_{\delta'} \ .$$

A contraction by $P^{\lambda'}_{\beta'}E^{\delta'}_{a}$ gives

$$P^{i}_{b}B^{\lambda'}_{i}\,(\delta^{b}_{a} - N_{a}N^{b}_{\mu}) = P^{\lambda'}_{\delta'}B^{\delta'}_{a} \ .$$

Now it is possible formulate

THEOREM 6. *Condition*

$$P^{i}_{b}B^{\delta'}_{i}N^{b}_{\mu} = 0 \tag{3.2}$$

is necessary and sufficient for coefficients ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$, *which are given in* (3.1), *to be the coefficients of connection. In this case the formula*

$$ {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = {}'\Gamma^{a}_{bc}B^{\beta}_{a}B^{b}_{\delta}B^{c}_{\gamma} + B^{\beta}_{a}B^{a}_{\delta\gamma} \ , \quad B^{a}_{\delta\gamma} := \frac{\partial}{\partial u^{\gamma}}B^{a}_{\delta} \tag{3.3}$$ *holds.*

Substituting (3.3) in (1.4) we get

$$\left(\frac{\partial B^i_\beta}{\partial u^\gamma} + {}'\Gamma^i_{sk} B^s_\beta B^k_\gamma\right) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = B^i_\alpha\left({}'\Gamma^a_{bc} B^\alpha_a B^b_\beta B^c_\gamma + B^\alpha_a B^a_{\beta\gamma}\right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Using (0.13) it follows that

$$(3.4) \qquad N^i_\mu N^\mu_a\left({}'\Gamma^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}\right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0 .$$

This relation is now stronger than condition (1.4), and it holds.

THEOREM 7. *Condition (3.4) is sufficient, but not necessary for the autoparallel curve of the contravariant type of subspace* $R\text{-}O_m$ *to be at the same time the autoparallel curve of the contravariant type of the embedding* $R\text{-}O_n$ *space.*

P r o o f. Let the curve $C: u^\alpha = u^\alpha(s)$ be the autoparallel curve of $R\text{-}O_m$. Then substituting d^2u/ds^2 from (1.1) in (1.3) and using (3.3) we get

$$\frac{d^2x^i}{ds^2} = \frac{\partial B^i_\alpha}{\partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} - B^i_\alpha B^\alpha_a\left({}'\Gamma^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}\right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

According to relation (0.13) and $\frac{dx^j}{ds} = B^j_\alpha \frac{du^\alpha}{ds}$ we get

$$\frac{d^2x^i}{ds^2} = -{}'\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + N^i_\mu N^\mu_a\left({}'\Gamma^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}\right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Applying (3.4) we get that the observed autoparallel curve of the subspace is at the same time the autoparallel curve of the contravariant type in the embedding $R\text{-}O_n$ space.

Now we shall consider the inverse question. Let the given curve be an autoparallel curve of the contravariant type in $R\text{-}O_n$, i.e.

$$(3.5) \qquad \frac{d^2x^i}{ds^2} + {}'\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

hols. Multiplying (3.3) by $\frac{du^\delta}{ds}\frac{du^\gamma}{ds}$ and using $\frac{dx^j}{ds} = B^j_\alpha\frac{du^\alpha}{ds}$ we get

$$'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} = '\Gamma^{a}_{bc}B^{\beta}_{a}\frac{dx^{b}}{ds}\frac{dx^{c}}{ds} + B^{\beta}_{a}B^{a}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} .$$

Since (3.5) holds we can eliminate $'\Gamma^{a}_{bc}\frac{dx^{b}}{ds}\frac{dx^{c}}{ds}$ and using (1.3) we finally get (1.1). So we can formulate

COROLLARY 1. *The observed autoparallel curve of the contravariant type of the embedding space* $R\text{-}O_n$ *without new conditions is an autoparallel curve of subspace* $R\text{-}O_m$*, if it belongs to this subspace.*

The stipulation of the above theorem follows directly from the relation (3.3), too. A contraction of (3.3) by $B^{k}_{\beta}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds}$ according to (0.13) gives

$$B^{k}_{\beta}\,'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} = \left[('\Gamma^{k}_{bc}B^{b}_{\delta}B^{c}_{\gamma} + B^{k}_{\delta\gamma}) - N^{k}_{\mu}N^{\mu}_{a}('\Gamma^{a}_{bc}B^{b}_{\delta}B^{c}_{\gamma}+B^{a}_{\delta\gamma})\right]\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds},$$

i.e. if $'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}$ has the form (3.3) it is not sufficient that (1.4) is satisfied, (3.4) must be satisfied too.

Further we shall consider autoparallel curves of the covariant type. For $''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ we use the formula

$$(3.6)\qquad ''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma} = ''\Gamma^{i}_{jk}B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma} + B^{\alpha}_{i}B^{i}_{\beta\gamma}$$

(see [1] (16)). The coefficients $''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ construced in this way satisfy the condition necessary and sufficient for the covariant differential of the metric tensor $G_{\alpha\beta}$ of the subspace to be zero. Substituting (3.6) in (2.4), using (0.13) we get

$$(3.7)\qquad N^{\mu}_{r}N^{i}_{\mu}(''\Gamma^{r}_{ak}B^{a}_{\alpha}B^{k}_{\beta} + B^{r}_{\alpha\beta})\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0 .$$

Now we can formulate

THEOREM 8. *Condition (3.7) is a sufficient, but not necessary condition to be the autoparallel curve of the covariant type of subspace* $R\text{-}O_m$ *at the same time the autoparallel curve of the covariant type of the embedding* $R\text{-}O_n$ *space.*

COROLLARY 2. *The observed autoparallel curve of embedding space* R-O$_n$ *without new conditions is an autoparallel curve of subspace* R-O$_m$, *if it belongs to this subspace.*

The proofs are analogous with the proofs given by the contravariant type.

Now we use the notation

$$"H^{\nu}_{\chi\alpha} := P^{\varepsilon}_{\alpha}P^{\nu}_{\sigma}("\Gamma^{s}_{jk}B^{j}_{\varepsilon}B^{k}_{\chi} + B^{s}_{\varepsilon\chi})N^{\delta}_{s}$$

given in [3] (3.5). Contraction by $Q^{\alpha}_{\lambda}Q^{\sigma}_{\nu}$ and substitution of the term which we got in (3.7) gives

$$N^{i}_{\mu}Q^{\varepsilon}_{\alpha}Q^{\mu}_{\nu}"H^{\nu}_{\beta\varepsilon}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0. \tag{3.8}$$

We suppose that in our subspace, (3.2) is satisfied and from this it follows that $N^{i}_{\mu}Q^{\mu}_{\nu} = N^{a}_{\nu}Q^{i}_{a}$ (see [2] (1.8)).

Substituting it in (3.8) we get $Q^{\varepsilon}_{\alpha}Q^{i}_{a}N^{a}_{\nu}"H^{\nu}_{\beta\varepsilon}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0.$ As it was proved in [3] $N^{a}_{\nu}"H^{\nu}_{\beta\varepsilon} = "\overset{*}{\nabla}_{\beta}B^{a}_{\varepsilon}$. It is known that in Otsuki's spaces $Q^{\varepsilon}_{\alpha}Q^{i}_{a}"\overset{*}{\nabla}_{\beta}B^{a}_{\varepsilon} = B^{i}_{\alpha\|\beta}$ holds and finally (3.8) has the form

$$B^{i}_{\alpha\|\beta}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0 \quad \text{or} \quad \frac{\bar{D}B^{i}_{\alpha}}{ds}\frac{du^{\alpha}}{ds} = 0 .$$

4. SPECIAL CASES OF A SUBSPACE WITH AN INDUCED CONNECTION

The induced connection of the subspace R-O$_m$ can be determined in various ways. For example

A/ If we suppose that for the covariant vectors ξ_{α} of the subspace R-O$_m$ satisfying $\xi_i = B^{\alpha}_{i}\xi_{\alpha}$ we can define the invariant differential by

$$\tilde{D}\xi_{\alpha} := B^{i}_{\alpha}D\xi_{i} \tag{4.1}$$

then the coefficients of connection $"\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ have a form like coefficients of connection $"\overset{*}{\Gamma}^{\alpha}_{\beta\gamma}$ of the intrinsic connection which

are given in (3.6). Using Otsuki's relation to determine the coefficients of connection ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and substituting ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and ${}''\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ in $(\tilde{1.4})$ and $(\tilde{2.4})$ we get results which are the same as in the former paragraph. With $(\tilde{1.4})$, $(\tilde{2.4})$ and $(\tilde{\tilde{1.4}})$, $(\tilde{\tilde{2.4}})$ we shall quote the equations we get from (1.4), (2.4) if in place of ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$, ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ we use ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, ${}''\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$, ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ respectively .

B/ If we suppose that for the contravariant vectors of the subspace satisfying $\xi^i = B^i_\alpha \xi^\alpha$ we define the covariant differential by

$$\tilde{\tilde{D}}\xi^\alpha := B^\alpha_i D\xi^i \tag{4.2}$$

then we get the coefficients of connection ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ in the form

$${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma} = {}'\Gamma^i_{jk} B^\alpha_i B^j_\beta B^k_\gamma + B^\alpha_i B^i_{\beta\gamma} \tag{4.3}$$

([1] (26) and [2] (1.1)). This is equivalent to (3.3), and the contravariant case coincides with that observed in the former paragraph.

The coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ in the form

$${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma} = P^r_j B^j_\lambda \overset{*}{Q}{}^\lambda_\beta ({}''\Gamma^a_{rk} B^\alpha_a B^k_\gamma - B^\alpha_{r\gamma}) + P^a_b B^\alpha_a N^b_\mu ({}''\Gamma^s_{jk} B^j_\lambda B^k_\gamma - B^s_{\lambda\gamma}) N^\mu_s \overset{*}{Q}{}^\lambda_\beta \tag{4.4}$$

we get from P^α_β and ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ using Otsuki's relation. Now the tensor P^α_β and the coefficients of connections ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ and ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ satisfy Otsuki's relation. Since we study the Riemann-Otsuki subspaces it must be that $\tilde{\tilde{D}}G_{\alpha\beta} = 0$. In [1] it was proved that form (3.6) of the coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ is necessary and sufficient for the metric tensor of the subspace to be a covariant constant. This means that the coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ from (4.4) can be used only in the special case in which (4.4) reduces on (3.6). But in these cases, for the auroparallel curves of covariant type the same holds as in the former paragraph for the curves of that type.

C/ If we suppose that for the covariant and contravariant vectors of the subspace R-O_m satisfying $\xi_i = B^{\alpha}_{i}\xi_{\alpha}$ and $\xi^i = B^{i}_{\alpha}\xi^{\alpha}$ respectively the invariant differential is defined by (4.1) and (4.2) respectively, we get the coefficients of connections defined by (3.6) and (4.3) (see [1] (26) and [2] (1.1)). In this case the coefficients of connection $'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and $''\tilde{\tilde{\Gamma}}^{\alpha}_{\beta\gamma}$ and the tensor P^{α}_{β} must satisfy Otsuki's relation, or as was proved in [2] it must be

$$(4.5)\quad P^{i}_{r}B^{\alpha}_{i}N^{t}_{\mu}('\Gamma^{a}_{sk}B^{s}_{\beta}B^{k}_{\gamma} + B^{a}_{\beta\gamma})N^{\mu}_{a} - P^{i}_{b}B^{b}_{\beta}N^{\mu}_{i}(''\Gamma^{s}_{rk}B^{\alpha}_{s}B^{k}_{\gamma} + B^{\alpha}_{r\gamma})N^{r}_{\mu} = 0.$$

Relation (4.5) is sufficient for the subspace of the R-O_n space to be a Riemann-Otsuki space with the coefficients of connection $'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and $''\tilde{\tilde{\Gamma}}^{\alpha}_{\beta\gamma}$ and the basic tensor P^{α}_{β}. From the above observation it obviously follows that the autoparallel curves of the co- or contravariant type of subspace R-O_m are at the same time the autoparallel curves of the embedding space, if (3.7) and (3.4) are satisfied. Inversely the autoparallel curve of the co- or contravariant type of the embedding space R-O_n is at the same time the autoparallel curve of the observed type of subspace R-O_m if it belongs to this subspace.

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REZIME

AUTOPARALELNE KRIVE RIEMANN-OTSUKIJEVIH PROSTORA

U radu su posmatrane autoparalelne krive potprostora Otsukijevog prostora. Odredjeni su uslovi pod kojima su te krive autoparalelne krive i u okolnom prostoru. Pošto su koeficijenti koneksije ko- i kontravarijantnog dela koneksije posmatranih prostora različiti posebno ispitujemo autoparalelne krive kovarijantnog tipa i autoparalele kontravarijantnog tipa.

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ON THE SPECTRUM OF MENDELSOHN n-TUPLE SYSTEMS

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ABSTRACT

In [6] Mendelsohn n-tuple systems (MnSs), which represent a generalization of Mendelsohn triple systems, were introduced and the spectrum of such systems investigated. In this paper we obtain some new results on the spectrum of MnSs using the results from [7].

1° In [3] Mendelsohn introduced a generalization of Steiner triple systems which he called cyclic triple systems. Such systems are now called Mendelsohn triple systems (MTSs). A cyclic triple is a collection t of three ordered pairs, none of which has equal coordinates, such that an element occurs as a first coordinate of an ordered pair iff it occurs as a second coordinate of an ordered pair in t. A MTS is a pair (S,T) where S is a finite nonempty set and T is a collection of cyclic triples of elements of S, such that every ordered pair of distinct elements of S is contained in exactly one triple of T. The number $|S|$ is called the order of (S,T). The spectrum of MTSs is the set of all integers

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$q > 1$ such that $q \not\equiv 2 \pmod 3$ except $q = 6$. A MTS is equivalent to a quasigroup satisfying the identities $(xy)x = y$ (semisymmetric) and $x^2 = x$ (idempotent).

In [6] a generalization of MTSs called Mendelsohn n-tuple systems was defined.

Let S be a finite nonempty set, $n \geq 3$. A cyclic n-tuple is the set

$$\{(x_1^{n-1}), (x_2^n), (x_3^n, x_1), \ldots, (x_n, x_1^{n-2})\}$$

of n distinct ordered (n-1)-tuples of elements of S, among which there is no (n-1)-tuple the coordinates of which are all equal. (By x_m^n we denote the sequence $x_m, x_{m+1}, \ldots, x_n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots \ldots, x$ (n times) will be denoted by $\overset{n}{x}$.)

A Mendelsohn n-tuple system (MnS), $n \geq 3$, is a pair (S,T), where S is a finite nonempty set and t is a collection of cyclic n-tuples of elements of S, such that every ordered (n-1)-tuple of elements of S, the coordinates of which are not all equal, belongs to exactly one cyclic n-tuple of T. The number $|S|$ is called the order of the MnS (S,T).

In [6] the spectrum of MnSs for different values of n was considered. It was proved that if n and q are even numbers, then there is no MnS of order q. The spectrum of M4S was determined to be the set of all odd integers greater than 1 and the spectrum of M5S was also investigated and some properties of MnS described. In this paper we obtain some new results on the spectrum of MnS using the results from [7].

2° An n-ary groupoid (n-groupoid) (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in N = \{1, \ldots, n\}$.

An n-quasigroup is called idempotent iff for every $x \in Q$ $A(\overset{n}{x}) = x$.

An n-quasigroup (Q,A) is called cyclic iff it satisfies the identity

$$A(A(x_1^n),x_1^{n-1}) = x_n .$$

An n-quasigroup is cyclic iff for every $i \in N_n$ and all $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \iff A(x_{i+1}^n, x_1^{i-1}) = x_i .$$

Cyclic n-quasigroups are a generalization of semisymmetric quasigroups.

A quasigroup $(Q,\cdot)$ is said to be self-orthogonal iff for every pair $(a,b) \in Q^2$ the system $xy = a$, $yx = b$ has a unique solution. An n-quasigroup (Q,A) is called self-orthogonal iff for every $(a_1^n) \in Q^n$ there exists a unique $(b_1^n) \in Q^n$ such that $A_i(b_1^n) = a_i$, $i=1,\dots,n$, where $A_1 = A$ and A_i are defined by $A_i(x_1^n) = A(x_i^n, x_1^{i-1})$, $i=2,\dots,n$. In [1],[2],[4],[5], the spectrum of self-orthogonal semisymmetric quasigroups (SOSQs) was investigated and in [7] the spectrum of self-orthogonal cyclic n-quasigroups (SOCnQs), which generalize the concept of SOSQs to higher dimensions, was considered.

3° In [6] the following relation between idempotent cyclic n-quasigroups and M(n+1)S was established:

Let $n+1$ be a prime, then there exists an idempotent cyclic n-quasigroup of order q iff there exists a M(n+1)S of order q.

Now we shall prove that every finite SOCnQ is necessarily idempotent.

THEOREM 1. *If* (Q,A) *is a finite* SOCnQ, *then* (Q,A) *is idempotent.*

P r o o f. Let (Q,A) be a finite SOCnQ and $a \in Q$ an arbitrary element. Then $A(\overset{n}{a}) = b$ implies $A_i(\overset{n}{a}) = b$ for all $i=2,\dots,n$, where A_i are defined by $A_i(x_1^n) = A(x_i^n, x_1^{i-1})$, $i=2,\dots,n$. Hence the ordered n-tuple $(\overset{n}{a})$ is a solution of the

system

$$(1) \qquad A_i(x_1^n) = b, \quad i=1,\dots,n\ ,$$

where $A_1 = A$. Since (Q,A) is self-orthogonal, the solution $(\overset{n}{a})$ of system (1) is unique.

This holds for all $a \in Q$, hence for every $a \in Q$ there is an element $b \in Q$ such that $(\overset{n}{a})$ is the unique solution of system (1). The mapping $f : a \mapsto b$ is obviously injective and since Q is finite, it is a bijection.

Let $a \in Q$ be an arbitrary element and $A(\overset{n}{a}) = b$. If $f^{-1}(a) = c$, then $A(\overset{n}{c}) = a$. A is cyclic which implies

$$A(b,\overset{n-1}{a}) = a, \quad A(a,b,\overset{n-2}{a}) = a, \dots, A(\overset{n-1}{a},b) = a\ .$$

Hence

$$A_i(\overset{n}{c}) = A_i(b,\overset{n-1}{a}), \quad i=1,\dots,n\ ,$$

and from the self-orthogonality of A it follows that $a=b=c$. So, we have proved that for every $a \in Q$ $A(\overset{n}{a}) = a$.

From the preceding theorem and the quoted connection between idempotent cyclic n-quasigroups and M(n+1)Ss, we get the result that if $n+1$ is prime then every SOCnQ of order q defines a M(n+1)S of the same order. Hence, using the results on the spectrum of SOCnQ obtained in [7], we get the following two theorems.

THEOREM 2. *Let* $n \geq 3$ *be prime,* $p_1,\dots,p_m$ *primes and* $k_1,\dots,k_m$ *positive integers such that* $p_i^{k_i} \equiv 1 \pmod n$, $i=1,\dots,m$. *Then for arbitrary non-negative integers* α_i, $i=1,\dots,m$, *there exists a* MnS *of order*

$$q = p_1^{k_1\alpha_1} \cdots p_m^{k_m\alpha_m}\ .$$

THEOREM 3. *Let* $n \geq 3$ *be prime and* $p_1,\dots,p_m$ *primes such that* $p_i > n$, $i=1,\dots,m$. *Then there are positive integers*

$s_1,\dots,s_m$, $1 \le s_i \le n-1$, $i=1,\dots,m$, *such that for all positive integers* α_i, $i=1,\dots,m$, *there is a* MnS *of order*

$$q = p_1^{s_1\alpha_1} \dots p_m^{s_m\alpha_m} .$$

REMARK. In some cases the values of s_i in the preceding theorem can be determined as degrees of irreducible factors of the polynomials which are defined in [7].

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REZIME

O SPEKTRU MENDELSONOVIH SISTEMA n-TORKI

Mendelsonovi sistemi n-torki, koji predstavljaju generalizaciju Mendelsonovih sistema trojki, su definisani i odredjene su neke vrednosti iz njihovog spektra u [6]. U ovom radu dobijeni su novi rezultati o spektru Mendelsonovih sistema n-torki koristeći rezultate iz [7].

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PARASTROPHY INVARIANT n-QUASIGROUPS

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ABSTRACT

An n-quasigroup (Q,A) is called a G-n-quasigroup iff $A=A^\sigma$ for all $\sigma\in G$, where G is a subgroup of the symmetric group of degree n+1 and A^σ is defined by:

$$A^\sigma(x_{\sigma 1},\dots,x_{\sigma n})=x_{\sigma(n+1)} \text{ iff } A(x_1,\dots,x_n)=x_{n+1}.$$

In the paper G-n-quasigroups are considered, and some of their properties described.

1° First we shall give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1},\dots,x_n$ will be denoted by $\{x_i\}_{i=m}^n$ or by x_m^n. If $m>n$, then x_m^n will be considered empty.

An n-ary groupoid (n-groupoid) (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1},x,a_{i+1}^n)=b$ has a unique solution x for every a_1^n, $b\in Q$, and every $i\in N_n=\{1,\dots,n\}$.

An n-quasigroup (Q,A) is isotopic to an n-quasigroup (Q,B) iff there exists a sequence $T=(\alpha_1^{n+1})$ of permutations of Q such that the following identity

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$$B(x_1^n) = \alpha_{n+1}^{-1} A(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, B is an isotope of A, and by $A^T = B$ we denote that A is isotopic to B by T. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma \in S_{n+1}$, where S_{n+1} is the symmetric group of degree n+1, then the n-quasigroup A^σ defined by

$$A^\sigma(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff A(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of A. If $\sigma, \tau \in S_{n+1}$, then $(A^\sigma)^\tau = A^{\sigma\tau}$ and

$$A(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff A^\tau(\{x_{\sigma\tau i}\}_{i=1}^n) = x_{\sigma\tau(n+1)}.$$

If $T = (\alpha_1^{n+1})$ is an isotopism of A, then $(A^T)^\sigma = (A^\sigma)^{T^\sigma}$,

where $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma \in S_{n+1}$ such that $A = A^\sigma$, then σ is called an autoparastrophism of A. The set of all autoparastrophisms of A is a subgroup of S_{n+1} which will be denoted by $\Pi(A)$.

An n-quasigroup (Q,A) is called cyclic [3] iff for every $i \in N_n$ and all $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \iff A(x_{i+1}^{n+1}, x_1^{i-1}) = x_i.$$

2° DEFINITION 1. *If* (Q,A) *is an* n-*quasigroup and* G *is a subgroup of* S_{n+1} *such that* $A = A^\sigma$ *for every* $\sigma \in G$, *then* A *is called a* G-n-*quasigroup*.

It is obvious that an n-quasigroup (Q,A) is a G-n-quasigroup iff $A = A^\sigma$ for all $\sigma \in \Gamma$, where Γ is a set of generators of the group G.

Some examples of G-n-quasigroups are:

1. Totally symmetric n-quasigroups are G-n-quasigroups with $G = S_{n+1}$.

2. Cyclic n-quasigroups, investigated in [3], are G-n-quasigroups, where G is the cyclic group generated by the cycle $(12\ldots n+1)$.

3. In [2] D.G.Hoffman has given a construction of a G-n-quasigroup (Q,A) of order mp, for every $m>n$, $p\geq 2$, and every subgroup $G \subseteq S_{n+1}$, such that $\Pi(A) = G$.

4. Let $(Q,+)$ be an Abelian group such that $x+x \neq 0$ for every $x \neq 0$. If a ternary operation A is defined by

$$A(x_1, x_2, x_3) = x_1 + x_2 - x_3,$$

then (Q,A) is a G-3-quasigroup, where G is Klein's four-group $\{(1),(12)(34),(13)(24),(14)(23)\}$. It is easy to see that A is neither totally symmetric nor cyclic, and that there exist such G-3-quasigroups of every order > 2.

From the definition of a parastrophe, we get the following proposition.

Let (Q,A) be an n-quasigroup and $\sigma \in S_{n+1}$, $\sigma i = n+1$. $A = A^{\sigma}$ iff for all $x_1^n \in Q$

$$A(x_{\sigma 1}, \ldots, x_{\sigma(i-1)}, A(x_1^n), x_{\sigma(i+1)}, \ldots, x_{\sigma n}) = x_{\sigma(n+1)} .$$

Consequently, every G-n-quasigroup can be defined as an n-quasigroup satisfying a system of identities.

PROPOSITION 1. *Let* (Q,A) *be an n-quasigroup and* G *a subgroup of* S_{n+1}. *A is a* G-n-*quasigroup iff for all* $\sigma \in \Gamma$ *and all* $x_1^n \in Q$

$$A(x_{\sigma 1}, \ldots, x_{\sigma(i-1)}, A(x_1^n), x_{\sigma(i+1)}, \ldots, x_{\sigma n}) = x_{\sigma(n+1)},$$

where Γ *is a set of generators of* G *and* $i = \sigma^{-1}(n+1)$.

In the preceding proposition, of course, Γ can be replaced by G.

From Proposition 1 it follows that the direct product of G-n-quasigroups is a G-n-quasigroup, which gives the possibility of constructing new G-n-quasigroups from the given ones.

PROPOSITION 2. *If* (Q,A) *is a* G-n-*quasigroup and* $\tau \in S_{n+1}$ *is such that the group* G *is invariant under the inner automorphism induced by* τ, *then* A^{τ} *is a* G-n-*quasigroup.*

P r o o f. Since G is invariant under the automorphism $\sigma \mapsto \tau\sigma\tau^{-1}$, it follows that for every $\sigma_i \in G$ there exists $\sigma_j \in G$ such that $\tau\sigma_i\tau^{-1} = \sigma_j$. Hence $A^{\tau\sigma_i\tau^{-1}} = A^{\sigma_j} = A$, and $(A^{\tau})^{\sigma_i} = A^{\tau}$ for all $\sigma_i \in G$, which means that A^{τ} is a G-n-quasigroup.

COROLLARY. *If* A *is a* G-n-*quasigroup and* G *a normal subgroup of a group* $G_1 \subseteq S_{n+1}$, *then every parastrophe* $A^{\tau}, \tau \in G_1$, *is also a* G-n-*quasigroup.*

PROPOSITION 3. *Let* (Q,A) *be a* G-n-*quasigroup, where* $G=\Pi(A)$. *A parastrophe* A^{τ} *is a* G-n-*quasigroup iff* G *is invariant under the inner automorphisam induced by* τ.

P r o o f. If $\tau\sigma\tau^{-1}=G$, then from Proposition 2 it follows that A^{τ} is a G-n-quasigroup.

Conversely, let A^{τ} be a G-n-quasigroup. Then for all $\sigma_i \in G$ $(A^{\tau})^{\sigma_i} = A^{\tau}$, that is, $A^{\tau\sigma_i\tau^{-1}} = A$. Since $G = \Pi(A)$, the only parastrophes which are equal to A are parastrophes induced by permutations from G. Hence $\tau\sigma_i\tau^{-1} \in G$ for all $\sigma_i \in G$.

Now we shall consider isotopes of G-n-quasigroups.

THEOREM 1. *Let an* n-*quasigroup* A *be isotopic to a* G-n-*quasigroup* B. *Then* A *is isotopic to the parastrophe* A^{σ} *for every* $\sigma \in G$, *and every parastrophe* A^{τ}, *where* τ *is a permutation such that* G *is invariant under the inner automorphism induced by* τ, *is an isotope of a* G-n-*quasigroup.*

P r o o f. B is a G-n-quasigroup, hence $B=B^{\sigma}$ for all $\sigma \in G$. Since the corresponding parastrophes of isotopic n-quasigroups are isotopic, it follows that A^{σ} is isotopic to $B = B^{\sigma}$ for all $\sigma \in G$.

If τ is such that G is invariant under the inner automorphism induced by τ, then Proposition 2 implies that B^{τ} is a G-n-quasigroup. Hence the corresponding parastrophe A^{τ} of A is an isotope of the G-n-quasigroup B^{τ}.

THEOREM 2. *Let* (Q,A) *be an n-quasigroup isotopic to its parastrophe* A^{σ} *by an isotopism* $T=(\alpha_1^{n+1})$, $A^T = A^{\sigma}$, *where* $\sigma \in S_{n+1}$, *and let* $(i_1 \ldots i_r)(j_1 \ldots j_s)\ldots(k_1 \ldots k_t)$ *be the decomposition of* σ *into* v *disjoint cycles (where the cycles of length* 1 *are included). Then there exist permutations* $\theta_1, \ldots \theta_v$ *of the set* Q *and an n-quasigroup* (Q,B) *which is isotopic to* A, *such that* B *is isotopic to* B^{σ} *by the isotopism*

$$(1,\ldots,1,\theta_1^{-1}\alpha_{i_1}\alpha_{\sigma(i_1)}\cdots\alpha_{\sigma^{r-1}(i_1)}\theta_1,1,\ldots$$
$$\ldots,1,\theta_2^{-1}\alpha_{j_1}\alpha_{\sigma(j_1)}\cdots\alpha_{\sigma^{s-1}(j_1)}\theta_2,1,\ldots$$
$$\ldots,1,\theta_v^{-1}\alpha_{k_1}\alpha_{\sigma(k_1)}\cdots\alpha_{\sigma^{t-1}(k_1)}\theta_v,1,\ldots,1),$$

where there are at least n+1-v *identity components, and at most one nonidentity component for every cycle of* σ. *The nonidentity component which corresponds to the cycle* $(i_1,\ldots,i_r)$ *can be at any of the places* $i_1,\ldots,i_r$ *and analogously for other cycles. If* (i_1) *is a cycle of length* 1, *then the corresponding nonidentity component is* $\theta_1^{-1}\alpha_{i_1}\theta_1$.

P r o o f. As in Theorem 5 from [3], let B be an arbitrary isotope of A, $B = A^S$, $S=(\beta_1^{n+1})$, and since $A^T = A^{\sigma}$ we have

$$(((B^{S^{-1}})^T)^{\sigma^{-1}})^S = B$$

which implies $B^{S^{-1}TS^{\sigma}} = B^{\sigma}$, where $S^{-1}TS^{\sigma} = (\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n})$.

If we put

$$(1_1)\qquad \beta_{i_2}^{-1}\alpha_{i_2}\beta_{\sigma(i_2)}=1,\dots,\beta_{i_r}^{-1}\alpha_{i_r}\beta_{\sigma(i_r)}=1,$$

$$(1_2)\qquad \beta_{j_2}^{-1}\alpha_{j_2}\beta_{\sigma(j_2)}=1,\dots,\beta_{j_s}^{-1}\alpha_{j_s}\beta_{\sigma(j_s)}=1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(1_v)\qquad \beta_{k_2}^{-1}\alpha_{k_2}\beta_{\sigma(k_2)}=1,\dots,\beta_{k_t}^{-1}\alpha_{k_t}\beta_{\sigma(k_t)}=1,$$

and take $\beta_{i_1},\beta_{j_1},\dots,\beta_{k_1}$ to be arbitrary permutations of Ω, then solving the systems (1_1), $(1_2),\dots,(1_v)$ as it is done in [3], Theorem 5, the theorem follows.

THEOREM 3. *If an* n-*quasigroup* (Ω,A) *is isotopic to an* n-*quasigroup* B *which coincides with one of its parastpophes,* $B=B^\sigma$, *then* A *is isotopic to* A^σ *by an isotopism* $T=(\alpha_1^{n+1})$ *such that*

$$\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}=1$$

for every cycle $(i_1\dots i_r)$ *in the decomposition of* σ *into disjoint cycles including cycles of length 1 (where, if* (j) *is a cycle of length 1, then* $\alpha_j=1$).

P r o o f. Let A be isotopic to n-quasigroup B, such that $B=B^\sigma$, by an isotopism $S=(\beta_1^{n+1})$, $A^S=B$. Then $A^S=B=B^\sigma=(A^S)^\sigma=(A^\sigma)^{S^\sigma}$, so $A^{S(S^\sigma)^{-1}}=A^\sigma$. As in [3], Theorem 6, denote $T=S(S^\sigma)^{-1}=(\alpha_1^{n+1})$ which implies

$$S^{-1}TS^\sigma=I=(1,\dots,1),$$

that is

$$(2)\qquad \beta_k^{-1}\alpha_k\beta_{\sigma k}=1,\quad k=1,\dots,n+1.$$

If we solve the subsystems of (2) which correspond to the disjoint cycle decomposition of σ separately, as it is done in the preceding theorem, we shall have for the cycle $(i_1\dots i_r)$

$$\beta_{i_1}^{-1}\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}\beta_{i_1}=1,$$

or $\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}=1$, which completes the proof.

REMARK. Theorems 2 and 3 generalize some results on cyclic n-quasigroups from [3].

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REZIME

PARASTROFNO INVARIJANTNE n-KVAZIGRUPE

n-kvazigrupa (Q,A) se naziva G-n-kvazigrupa ako i samo ako je $A = A^{\sigma}$ za svako $\sigma \in G$, gde je G podgrupa simetrične grupe stepena $n+1$, a A^{σ} je definisana sa: $A^{\sigma}(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)}$ ako i samo ako je $A(x_1,\dots,x_n) = x_{n+1}$.

U ovom radu razmatrane su G-n-kvazigrupe i odredjena neka njihova svojstva.

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ALTERNATING SYMMETRIC n-QUASIGROUPS

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ABSTRACT

Alternating symmetric (AS) n-quasigroups are defined and considered. An n-quasigroup (Q,f) is caled an AS-n-quasigroup iff $f(x_1,\dots,x_n) = x_{n+1} \iff f(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)}$ for every even permutation σ of the set $\{1,\dots,n+1\}$. AS-n-quasigroups represent a generalization of semisymmetric quasigroups. Several equivalent definitions of an AS-n-quasigroup are given and it is proved that every AS-n-quasigroup, $n > 3$, defines a family of totally symmetric (n-2)-quasigroups. Some properties of (i,j)-associative AS-n-quasigroups are determined and full characterization of AS-n-groups is given. Autotopisms and isotopism of AS-n-quasigroups are considered. Necessary and sufficient conditions for a principal isotope of an AS-n-quasigroup to be an AS-n-quasigroup are given.

1^{o}

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

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The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by $\{x_i\}_{i=m}^n$ or by x_m^n. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (n times) will be denoted by $\overset{n}{x}$. If $n \leq 0$, then $\overset{n}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q,f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in N_n = \{1, \ldots, n\}$.

An n-quasigroup (Q,f) is isotopic to an n-quasigroup (Q,g) iff there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, g is an isotope of f, and by $f^T = g$ we denote that f is isotopic to g by T. If α_{n+1} is the identity mapping, then g is said to be a principal isotope of f. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$. If T is an isotopism of (Q,f) to itself, that is, $f^T = f$, then T is called an autotopism of f.

By S_n we denote the symmetric group of degree n and by A_n its alternating subgroup.

If (Q,f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f. If $\sigma, \tau \in S_{n+1}$, then $(f^\sigma)^\tau = f^{\sigma\tau}$ and

$$f(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f^\tau(\{x_{\sigma\tau i}\}_{i=1}^n) = x_{\sigma\tau(n+1)} .$$

If $T = (\alpha_1^{n+1})$ is an isotopism of f, then $(f^T)^\sigma = (f^\sigma)^{T^\sigma}$, where $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$.

If (Q,f) is an n-quasigroup and $\sigma \in S_{n+1}$ such that $f = f^\sigma$, then σ is called an autoparastrophism of f. The set of all autoparastrophism of f is a subgroup of S_{n+1}

which will be denoted by $\Pi(f)$.

An n-quasigroup (Q,f) is called totally symmetric (TS) iff $f^\sigma = f$ for every $\sigma \in S_{n+1}$.

An n-quasigroup (Q,f) is called (i,j)-associative iff the following identity holds

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is (i,j)-associative for all $i,j \in N_n$ is called an n-group.

2°

DEFINITION. *An n-quasigroup* (Q,f) *is called alternating symmetric (AS) iff for every* $\sigma \in A_{n+1}$

$$f = f^\sigma .$$

It is obvious that an n-quasigroup (Q,f) is an AS-n-quasigroup iff $f = f^\sigma$ for all $\sigma \in \Gamma$, where Γ is a generating set of the group A_{n+1}.

From the definition it follows that every TS-n-quasigroup is also an AS-n-quasigroup. But there are AS-n-quasigroups which are not TS, which follows from [2] where D.G.Hoffman has proved that for every $m > n$, $p \geq 2$, and every subgroup $G \subseteq S_{n+1}$ there exists an n-quasigroup (Q,f) of order mp such that $\Pi(f) = G$.

When $n = 2$ from the definition it follows that a quasigroup (binary) $(Q,\cdot)$ is AS iff $(\cdot) = (\cdot)^{(123)} = (\cdot)^{(132)}$, i.e. iff $xy = z \Leftrightarrow yz = x \Leftrightarrow zx = y$. These equivalences imply that $(Q,\cdot)$ is an AS quasigroup iff the identities

(1) $$y(xy) = x , \qquad (xy)x = y$$

hold.

A quasigroup satisfying the identities (1) is called semisymmetric, so binary AS quasigroups are in fact semisymmetric quasigroups. In [5] so-called cyclic n-quasigroups where introduced and such n-quasigroups are another generalization

of semisymmetric quasigroups (an n-quasigroup (Q,f) is cyclic iff the identity $f(f(x_1^n),x_1^{n-1}) = x_n$ holds or equivalently iff $f = f^\sigma$, $\sigma = (1,2,\dots,n+1)$).

3°

AS-n-quasigroups can be described as n-quasigroups satisfying certain systems of identities. If (Q,f) is an n-quasigroup and $\sigma \in S_{n+1}$, $\sigma k = n+1$, then $f^\sigma = f$ iff for all $x_1^n \in Q$

$$f(\{x_{\sigma i}\}_{i=1}^{k-1}, f(x_1^n), \{x_{\sigma i}\}_{i=k+1}^{n}) = x_{\sigma(n+1)} \,.$$

So we have the following theorem.

THEOREM 1. *An n-quasigroup* (Q,f) *is an AS-n-quasigroup iff for every* $\sigma \in \Gamma$ *and all* $x_1^n \in Q$

$$f(\{x_{\sigma i}\}_{i=1}^{k-1}, f(x_1^n), \{x_{\sigma i}\}_{i=k+1}^{n}) = x_{\sigma(n+1)} \,,$$

where Γ *is a set of generators of* A_{n+1} *and* $k = \sigma^{-1}(n+1)$.

From Theorem 1 we get that the direct product of AS-n-quasigroups is also an AS-n-quasigroup and a subquasigroup of an AS-n-quasigroup is an AS-n-quasigroup.

Now we shall give explicitly some of the systems of identities defining AS-n-quasigroups. A well known generating set of A_n is $\Gamma = \{(123),(124),\dots,(12n)\}$, and from Theorem 1 we get the following corollary.

COROLLARY 1. *An n-quasigroup* (Q,f) *is AS iff at least one of the following (eqivalent) systems of identities holds*

$$(2) \qquad \begin{cases} f(x_2,x_i,x_3^{i-1},x_1,x_{i+1}^n) = f(x_1^n), \quad i = 3,\dots,n, \\ f(x_2,f(x_1^n),x_3^n) = x_1 \,, \end{cases}$$

$$\Gamma_1 = \{(123),(124),\dots,(1,2,n+1)\},$$

$$(3)\quad \begin{cases} f(f(x_1^n),x_2^{n-1},x_1) = x_n, \\ f(x_1,f(x_1^n),x_3^{n-1},x_2) = x_n, \\ f(x_1^2,f(x_1^n),x_4^{n-1},x_3) = x_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ f(x_1^{n-2},f(x_1^n),x_{n-1}) = x_n, \end{cases}$$

$$\Gamma_2 = \{(n+1,n,1),(n+1,n,2),\dots,(n+1,n,n-1)\},$$

$$(4)\quad \begin{cases} f(x_n,x_2^{n-1},f(x_1^n)) = x_1, \\ f(x_1,x_n,x_3^{n-1},f(x_1^n)) = x_2, \\ f(x_1^2,x_n,x_4^{n-1},f(x_1^n)) = x_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ f(x_1^{n-2},x_n,f(x_1^n)) = x_{n-1}, \end{cases}$$

$$\Gamma_3 = \{(n,n+1,1),(n,n+1,2),\dots,(n,n+1,n-1)\}.$$

If (Q,f) is an n-quasigroup, then (see [1]) a parastrofe f^{π_i}, where $i \in N_n$, is defined by

$$f^{\pi_i}(x_1^{i-1},x_{n+1},x_{i+1}^n) = x_i \iff f(x_1^n) = x_{n+1}\,.$$

The operation f^{π_i} is called i-th inverse operation for f. An AS-n-quasigroup can be defined also using inverse operations for f.

THEOREM 2. *An n-quasigroup* (Q,f) *is AS iff*

$$f^{\pi_i\pi_j} = f$$

for every $i,j \in N_n$.

The next theorem shows that for $n > 3$ every AS-n-quasigroup defines a family of TS-(n-2)-quasigroups.

THEOREM 3. *Let* (Q,f) *be an AS-n-quasigroup,* $n > 3$, *and* $a,b \in Q$ *arbitrary elements,* $i,j \in N_{n+1}$, $i \neq j$. *The (n-2)-quasigroup* (Q,f) *defined by*

$$g(x_1^{n-2}) = x_{n-1} \iff f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1}$$

is a TS-(n-2)-quasigroup.

P r o o f. Since the alternating group A_{n+1} is (n-1)-fold transitive permutation group, it follows that for each two ordered (n-1)-tuples of elements from N_{n+1} there exists a permutation from A_{n+1} which maps one of these (n-1)-tuples onto another. This means that the equality

$$f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1} ,$$

which has n-1 variables, remains valid if all variables are arbitrarily permuted (where a, b remain at their places), i.e. $f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1} \iff f(y_1^{i-1},a,y_i^{j-2},b,y_{j-1}^{n-2}) = y_{n-1}$, where (y_1^{n-1}) is an arbitrary permutation of (x_1^{n-1}). Hence

$$g(x_1^{n-2}) = x_{n-1} \iff g(y_1^{n-2}) = y_{n-1},$$

which means that g is TS.

4°

Since every cycle of odd length is an even permutation, we have the following proposition.

PROPOSITION 1. *If* n *is even, then every AS-n-quasigroup is a cyclic n-quasigroup.*

From the preceding proposition and the results obtained in [6] for (i,j)-associative cyclic n-quasigroups, we get the following two theorems.

THEOREM 4. *Let* (Q,f) *be an (i,j)-associative AS-n-quasigroup,* n *even. Then* f *is (i+m,j+m)-associative for every integer* m *(where* $(i+m,j+m)$ *is reduced modulo* n*).*

THEOREM 5. *Let* (Q,f) *be an* (i,j)*-associative AS-n-quasigroup,* n *even, where* $j-i$ *is relatively prime to* n*. Then* f *is an n-group.*

THEOREM 6. *Let* (Q,f) *be an* (i,j)*-associative AS-n-quasigroup,* n *odd. Then* f *is* $(i+m,j+m)$*-associative, where* m *is an arbitrary integer such that* $1 \le i+m \le n$, $1 \le j+m \le n$.

P r o o f. Let f be (i,j)-associative. Then for all $x_1^{2n-1} \in Q$

$$f(x_1^{i-1},f(x_i^{i+n-1}),x_{i+n}^{2n-1}) = f(x_1^{j-1},f(x_j^{j+n-1}),x_{j+n}^{2n-1}).$$

Let $i,j < n$. f is AS, hence $f = f^{\sigma}$, where $\sigma = (12\ldots n)$, that is

$$f(x_{2n-1},x_1^{i-1},f(x_i^{i+n-1}),x_{i+n}^{2n-2}) = f(x_{2n-1},x_1^{j-1},f(x_j^{j+n-1}),x_{j+n}^{2n-2}),$$

i.e. f is $(i+1,j+1)$-associative. If $1 < i,j$, by an analogous procedure, using σ^{-1} instead of σ, we get that f is $(i-1,j-1)$-associative.

Hence f is $(i+m,j+m)$-associative, where m is an arbitrary integer such that $1 \le i+m \le n$, $1 \le j+m \le n$.

THEOREM 7. *Let* (Q,f) *be an n-group. Then* (Q,f) *is AS iff there exists an Abelian group* $(Q,+)$ *such that* $x = -x$ *for all* $x \in Q$*, and*

$$f(x_1^n) = \sum_1^n x_i + c ,$$

where c *is a fixed element from* Q.

P r o o f. Let (Q,f) be an AS-n-group. Then by Hosszù theorem there exist a group $(Q,\cdot)$, its automorphism θ and an element $c \in Q$ such that

$$f(x_1^n) = x_1\theta x_2\theta^2 x_3\ldots\theta^{n-1}x_n c ,$$

where $\Theta c = c$ and for all $x \in Q$ $\Theta^{n-1}x = cxc^{-1}$. f is AS, hence $f = f^{\sigma}$, where $\sigma = (1,2,n+1)$, and the following identity is valid

$$f(x_2, f(x_1^n), x_3^n) = x_1,$$

that is

$$(5)\qquad x_2\Theta(x_1\Theta x_2\Theta^2x_3\ldots\Theta^{n-1}x_nc)\Theta^2x_3\ldots\Theta^{n-1}x_nc = x_1.$$

If we put in the preceding equality $x_i = e$, $i = 1,\ldots,n$, where e is the unit of $(Q,\cdot)$, we get that $c^2 = e$. Now putting in (5) $x_i = e$, $i = 2,\ldots,n$ it follows $\Theta x_1 = x_1$, i.e. Θ is the identity mapping of Q. If in (5) we put $x_i = e$, $i = 1,3,\ldots,n$, we obtain $\Theta^2x_2 = x_2^{-1}$ which means that for all $x \in Q$ $x = x^{-1}$. Hence $(Q,\cdot)$ is an Abelian group and

$$f(x_1^n) = x_1x_2\ldots x_nc.$$

The converse part of the theorem is obvious.

Since the group $(Q,\cdot)$ such that $x = x^{-1}$ for all $x \in Q$ is of order 2^t, $t \in N$, and for every $t \in N$ there exists such group of order 2^t, we have the following corollary.

COROLLARY 2. *There exists a nontrivial* finite AS-n-group* (Q,f) *of order* q *iff* $q = 2^t$, $t \in N$.

5°

PROPOSITION 2. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* (Q,f) *and* $\sigma \in A_{n+1}$, *then* $T^{\sigma} = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$ *is also an autotopism of* f.

P r o o f. Since T is an autotopism we have $f^T = f$, and since f is AS it follows $f^{\sigma} = f$. Hence $f = (f^T)^{\sigma} = (f^{\sigma})^{T^{\sigma}} = f^{T^{\sigma}}$, that is, T^{σ} is an autotopism of f.

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*) *An n-group* (Q,f) *is called trivial iff* $|Q| = 1$.

PROPOSITION 3. *Let* α,β *be permutations of a set* Q, $n > 2$. $(\alpha,\beta,\overset{n-1}{\varepsilon})$ *is an autotopism of an AS-n-quasigroup* (Q,f) *iff* $\beta = \alpha^{-1}$ *and for some* $i,j \in N_n$, $i \neq j$, *the identity*

$$f(x_1^{i-1},\alpha x_i,x_{i+1}^n) = f(x_1^{j-1},\alpha x_j,x_{j+1}^n) \tag{6}$$

holds (by ε *we denote the identity mapping of* Q*).*

P r o o f. Let $(\alpha,\beta,\overset{n-1}{\varepsilon})$ be an autotopism of (Q,f). By Proposition 2 for every $i,j \in N_n$, $i \neq j$, $(\overset{i-1}{\varepsilon},\alpha,\overset{n-1}{\varepsilon},\beta)$ and $(\overset{j-1}{\varepsilon},\alpha,\overset{n-j}{\varepsilon},\beta)$ are autotopism of f. Consequently

$$\beta f(x_1^n) = f(x_1^{i-1},\alpha x_i,x_{i+1}^n) = f(x_1^{j-1},\alpha x_j,x_{j+1}^n).$$

Putting in the preceding identity $\alpha^{-1}x_i$ instead of x_i, we get

$$f(x_1^n) = f(x_1^{i-1},\alpha^{-1}x_i,x_{i+1}^{j-1},\alpha x_j,x_{j+1}^n),$$

i.e. $(\overset{i-1}{\varepsilon},\alpha^{-1},\overset{j-i-1}{\varepsilon},\alpha,\overset{n-j+1}{\varepsilon})$ is an autotopism of f, which implies that $(\alpha,\alpha^{-1},\overset{n-1}{\varepsilon})$ is also an autotopism of f. Two autotopisms which differ in only one component must be equal, hence $\alpha = \alpha^{-1}$.

Now let the identity (6) holds for some $i,j \in N_n$, $i \neq j$. Putting in (6) $\alpha^{-1}x_i$ instead of x_i, we get similarly as in the preceding part of the proof, that $(\alpha,\alpha^{-1},\overset{n-1}{\varepsilon})$ is an autotopism of f.

REMARK. It is easy to see that if the identity (6) holds for some $i,j \in N_n$, $i \neq j$, then it holds for every such i,j.

Let (Q,f) be an n-quasigroup, $i \in N_n$. A permutation α of Q is said to be i-inverse regular for f iff $(\overset{i-1}{\varepsilon},\alpha,\overset{n-1}{\varepsilon},\alpha^{-i})$ is an autotopism of f. A permutation which is i-inverse regular for f for every $i \in N_n$ is called inverse regular for f. The set of all inverse regular permutations for f will be denoted by $V(f)$ (see [1]).

If (Q,f) is an AS-n-quasigroup, $n > 2$, then it is easy to see that if α is for some $i \in N_n$ i-inverse regular for f, then α is inverse regular for f.

Now we have the following corollary from Proposition 3.

COROLLARY 3. *If* $(\alpha,\beta,\overset{n-1}{\varepsilon})$ *is an autotopism of an AS-n-quasigroup* (Q,f), $n > 2$, *then* α *and* β *are inverse regular permutations for* f.

PROPOSITION 4. *Let* $T = (\alpha_1^{n+1})$ *be an autotopism of an AS-n-quasigroup* (Q,f).

If $n \geq 2$, *then for any* $i,j,k \in N_{n+1}$, $i \neq j \neq k \neq i$, $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_k,\alpha_k^{-1}\alpha_i,\overset{n-2}{\varepsilon})$ *or* $(\alpha_j^{-1}\alpha_k,\alpha_i^{-1}\alpha_j,\alpha_k^{-1}\alpha_i,\overset{n-2}{\varepsilon})$ *is an autotopism of* f.

If $n > 3$ *then for all* $i,j \in N_{n+1}$, $i \neq j$, $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_i,\overset{n-1}{\varepsilon})$ *is an autotopism of* f.

P r o o f. If $T = (\alpha_1^{n+1})$ is an autotopism of f, $n \geq 2$, then T^{-1} and T^{ϕ}, $\phi = (i,j,k)$, $i,j,k \in N_{n+1}$, $i \neq j \neq k \neq i$, are also autotopisms of f. Then

$$T^{-1}T^{\phi} = (\overset{i-1}{\varepsilon},\alpha_i^{-1}\alpha_j,\overset{j-i-1}{\varepsilon},\alpha_j^{-1}\alpha_k,\overset{k-j-1}{\varepsilon},\alpha_k^{-1}\alpha_i,\overset{n-k+1}{\varepsilon})$$

is also an autotopism of f.

If $\sigma = (3k)(2j)(1i)$ and $\tau = (12)(3k)(2j)(1i)$, then one of these two permutations is even. If σ is even, then $(T^{-1}T^{\phi})^{\sigma}$ is an autotopism of f, and if τ is even then $(T^{-1}T^{\phi})^{\tau}$ is an autotopism of f, hence the first part of the proposition is proved.

Now let $n > 3$ and let $S = (\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_k,\alpha_k^{-1}\alpha_i,\overset{n-2}{\varepsilon})$ be an autotopism of f (the proof is analogous if $(\alpha_j^{-1}\alpha_k,\alpha_i^{-1}\alpha_j,\alpha_k^{-1}\alpha_i,\overset{n-2}{\varepsilon})$ is an autotopism). Applying to S the first part of this proposition, taking the first and any two of the last $n-3$ components of S, we get that $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_i,\overset{n-1}{\varepsilon})$ is an autotopism of f.

PROPOSITION 5. *Let* $n > 3$. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* (Q,f), *then for every* $\sigma \in S_{n+1}$ $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$ *is also an autotopism of* f.

P r o o f. By the preceding proposition, for every $i,j \in N_{n+1}$, $i \neq j$, $S = (\alpha_i^{-1}\alpha_j, \alpha_j^{-1}\alpha_i, \overset{n-1}{\varepsilon})$ is an autotopism of f. Proposition 2 implies that

$$S^{(2j)(1i)(34)} = S^{(2j)(1i)} = (\overset{i-1}{\varepsilon}, \alpha_i^{-1}\alpha_j, \overset{j-i-1}{\varepsilon}, \alpha_j^{-1}\alpha_i, \overset{n-j+1}{\varepsilon})$$

is an autotopism of f. Multiplying this autotopism by T from the left, we get that

$$(\alpha_1^{i-1}, \alpha_j, \alpha_{i+1}^{j-1}, \alpha_i, \alpha_{j+1}^{n+1}) = T^{(ij)}$$

is an autotopism of f. We have obtained that for every transposition (ij) $T^{(ij)}$ is an autotopism of f, the set of all transpositions generates S_{n+1}, hence T^σ is an autotopism for every $\sigma \in S_{n+1}$.

For $n > 3$ some properties of AS-n-quasigroups are very close to the corresponding properties of TS-n-quasigroups. Now we shall give several propositions describing such properties, but we shall omit the proofs since they can be given on the basis of the proved theorems and propositions analogously to the corresponding proofs for TS-n-quasigroups (see [1],[4]).

PROPOSITION 6. *If* $A_i(f)$ *denotes the group of i-th components of all autotopisms of an AS-n-quasigroup* (Q,f), *then* $A_i(f)$ *coincides with* $A_j(f)$ *for all* $i,j \in N_{n+1}$. *This group we shall denote by* $A_o(f)$.

PROPOSITION 7. *If* $n > 3$ *and* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* (Q,f), *then* $\alpha_i^{-1}\alpha_j$ *is inverse regular for* f *for every* $i,j \in N_{n+1}$.

PROPOSITION 8. *Every autotopism* $T = (\alpha_1^{n+1})$ *of an AS-n-quasigroup* (Q,f), $n > 3$, *can be represented in the form*

$$T = \alpha(\lambda_1, \ldots, \lambda_n, \varepsilon) ,$$

where $\lambda_1, \ldots, \lambda_n \in V(f)$.

PROPOSITION 9. *If a component of an autotopism of an AS-n-quasigroup* (Q,f) *is* ε, $n > 3$, *then all other componets of that autotopism are inverse regular permutations for* f.

PROPOSITION 10. *If* (Q,f) *is an AS-n-quasigroup,* $n > 3$, *then the set* $V(f)$ *of all inverse regular permutations for* f *is an Abelian group.*

PROPOSITION 11. *Let* (Q,f) *be an AS-n-quasigroup,* $n > 3$. *If a component of an autotopism of* f *is inverse regular for* f, *then all components of that autotopism are inverse regular for* f.

PROPOSITION 12. *Let* (Q,f) *be an AS-n-quasigroup,* $n > 3$. *For every permutation* $\alpha \in A_o(f)$ *there exists exactly one permutation* $\phi \in V(f)$, *such that* $(\overset{n}{\alpha}, \alpha\phi)$ *is an autotopism of* f.

PROPOSITION 13. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* (Q,f) *such that* $\alpha_1^{n+1} \in V(f)$, *then*

$$\alpha_1\alpha_2\ldots\alpha_n = \varepsilon .$$

THEOREM 8. *A principal isotope* (Q,g) *of an AS-n-quasigroup* (Q,f), $n > 3$, $f^T = g$, $T = (\alpha_1^n, \varepsilon)$, *is an AS-n-quasigroup iff all components of* T *are inverse regular for* f.

P r o o f. Let (Q,g), $g = f^T$, be an AS-n-quasigroup. Then

$$g(x_2, g(x_1^n), x_3^n) = x_1, \tag{7}$$

which gives

$$f(\alpha_1 x_2, \alpha_2 f(\{\alpha_i x_i\}_{i=1}^n), \{\alpha_i x_i\}_{i=3}^n) = x_1 ,$$

and, since f as also AS-n-quasigroup, it follows

$$f(x_1,\alpha_1 x_2,\{\alpha_i x_i\}_{i=1}^n) = \alpha_2 f(\{\alpha_i x_i\}_{i=1}^n) .$$

Hence

$$f(x_1^n) = \alpha_2 f(\alpha_1 x_1,\alpha_2\alpha_1^{-1}x_2,x_3^n) ,$$

i.e. $(\alpha_1,\alpha_2\alpha_1^{-1},\overset{n-2}{\varepsilon},\alpha_2)$ is an autotopism of f.

By Proposition 9 α_1 and α_2 are inverse regular for f. Using instead of (7) other identities which are satisfied by every AS-n-quasigroup, we get by an analogous procedure that $\alpha_3,\dots,\alpha_n$ are also inverse regular for f.

Conversely, let now an n-quasigroup (Q,f) be isotopic to an AS-n-quasigroup (Q,f), $f^T = g$, $T = (\alpha_1^n,\varepsilon)$, where all components of T are inverse regular permutations for f.

Then

$$\alpha_n g(g(x_1^n),x_2^{n-1},x_1) = \alpha_n f(\alpha_1 f(\{\alpha_i x_i\}_{i=1}^n),\{\alpha_i x_i\}_{i=2}^{n-i},\alpha_n x_1) =$$

$$= f(f(x_1,\{\alpha_i x_i\}_{i=2}^n),\{\alpha_i x_i\}_{i=2}^{n-1},x_1) = f(f(y_1^n),y_2^{n-1},y_1) = y_n = \alpha_n x_n ,$$

where we have used that α_n and α_1 are inverse regular for f, that f is an AS-n-quasigroup and we have put $y_1 = x_1$, $y_i = \alpha_i x_i$, $i = 2,\dots,n$.

Hence we have obtained the identity

$$g(g(x_1^n),x_2^{n-1},x_1) = x_n .$$

One can prove analogously that g satisfies all other identities from (3), so (Q,g) is an AS-n-quasigroup.

REMARK. Since the second part of the proof of the preceding theorem is valid for every $n \geq 2$, it follows that for every such n a principal isotope of an AS-n-quasigroup (Q,f) is an AS-n-quasigroup if all components of the isotopy are inverse regular for f.

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REZIME

ALTERNATIVNO SIMETRIČNE n-KVAZIGRUPE

U radu su definisane i razmatrane alternativno simetrične (AS) n-kvazigrupe. n-kvazigrupa (Q,f) se naziva AS-n--kvazigrupa ako i samo ako za svaku parnu permutaciju σ skupa $\{1,\dots,n+1\}$ važi $f(x_1,\dots,x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)}$. AS-n-kvazigrupe predstavljaju jednu generalizaciju polusimetričnih kvazigrupa. Date su neke ekvivalentne definicije AS-n-kvazigrupa i dokazano da svaka AS-n-kvazigrupa, $n > 3$, definiše familiju totalno simetričnih $(n-2)$-kvazigrupa. Odredjene su neke osobine (i,j)-asocijativnih AS-n-kvazigrupa i data potpuna karakterizacija AS-n-grupa. Zatim su razmatrane autotopije i izotopije AS-n-kvazigrupa. Dati su potrebni i dovoljni uslovi da glavni izotop AS-n-kvazigrupa bude AS-n-kvazigrupa.

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ON (k,n,q) - NETS

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ABSTRACT

(k,n)-Nets, $n \in N\backslash\{1\}$, $k \in N\backslash\{1,\ldots,n\}$, represent a generalization of k-nets, $k \in N\backslash\{1,2\}$; namely (k,2)-nets are k-nets [8-9]. Finite (k,n)-nets of order $q \in N\backslash\{1\}$ are also called (k,n,q)-nets [7]. In this article a connection between k,n and q is established.

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Let T be a nonempty set of elements called *points*. Let a nonempty set B have as its elements some subsets of the set T, called *blocks*. Finally, let the sets $L_1,\ldots,L_k$, $k \in N\backslash\{1,\ldots,n\}$,

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be equivalence classes classifying the set $\mathcal{B}$. Then we say that $(T, L_1, \ldots, L_k)$ is a (k,n)-net iff the following properties hold:

nR1. The intersection of any n blocks belonging to different classes $L_{i_1}, \ldots, L_{i_n}$; $i_1, \ldots, i_n \in \{1, \ldots, k\}$ has exactly one element (point); and

nR2. Any point from T belongs to exactly one block of each class L_i, $i \in \{1, 2, \ldots, k\}$ [1].

EXAMPLE. By superimposing the pictures 1_1 and 1_2 we obtain the picture of a (4,3)-net of order 2. We have:

$T = \{A, B, C, D, A', B', C', D'\}$, $L_1 = \{ABCD$ [2] $A', B', C', D'\}$,

$L_2 = \{BB'CC', AA'DD'\}$, $L_3 = \{AA'\ BB', CC'DD'\}$, and

$L_4 = \{A'B\ C'D, AB'CD'\}$.

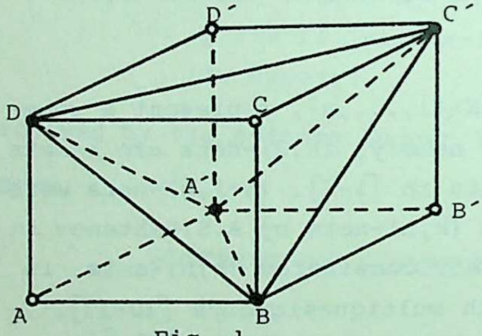

Fig. 1_1

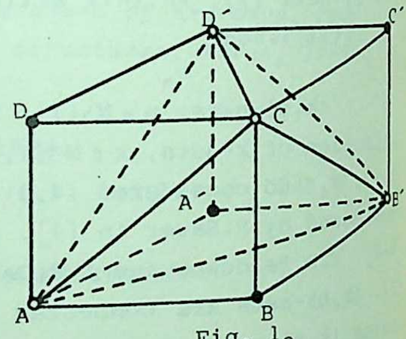

Fig. 1_2

Taking into account nR1, we can see that there does not exists a $(k,3,2)$-net for $k > 4$.

The following statement is known:

STATEMENT 1. All classes L_i, $i \in \{1, \ldots, k\}$, have the same number of blocks, and all blocks have the same number of points. Also, if $|L_i| = |Q|$ and $b_i \in L_i$, then $|b_i| = |Q^{n-1}|$.

The number of blocks in each class is called the order of (k,n)-net. A consequence of Statement 1 is:

---

1) In the description of (k,n)-nets, the incidence relation is usually used.

2) We write ABCD instead of $\{A,B,C,D\}$.

STATEMENT 2. If $(T, L_1, \dots, L_k)$ is a (k,n)-net of order $q \in N \setminus \{1\}$, then the number of points in each block is q^{n-1}.

We are going to prove the following theorem.

THEOREM 3. *If* $(T, L_1, \dots, L_k)$ *is a* (k,n)- *net of order* $q \in N \setminus \{1\}$, *then*

$$(k-1)\cdot \ldots \cdot (k-n+1) \leq (n-1)!\,q^{n-1} \tag{1}$$

REMARK. If n=2, (1) becomes $k-1 \leq q$, which is a known connection between k and q for k-nets of order q [8-9].

P r o o f. Let $b_i \in L_i$ and $T \notin b_i$ (Fig.2). There are exactly k blocks incident with T (nR2). Consider the blocks $b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_k$ from the classes $L_1, \dots, L_{i-1}, L_{i+1}, \dots, L_k$ respectively. Each unordered (n-1)-tuple of blocks from the set $\{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_k\}$ has exactly one common point with the block $b_i \in L_i$ (nR1).

Let

$$B_\alpha = \{b_{\alpha_1}, \dots, b_{\alpha_{n-1}}\}$$

and

$$B_\beta = \{b_{\beta_1}, \dots, b_{\beta_{n-1}}\}$$

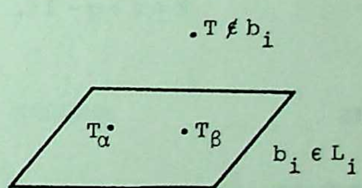

Fig. 2

be two such (n-1)-tuples; $|B_\alpha| = |B_\beta| = n-1$, and let

$$b_{\alpha_1} \cap \dots \cap b_{\alpha_{n-1}} \cap b_i = \{T_\alpha\} \in L_i \qquad \text{(Fig. 2) and}$$

$$b_{\beta_1} \cap \dots \cap b_{\beta_{n-1}} \cap b_i = \{T_\beta\} \in L_i \qquad \text{(Fig. 2)}$$

If $B_\alpha \neq B_\beta$, then $\max |B_\alpha \cup B_\beta| = 2n-2$ and $\min |B_\alpha \cup B_\beta| = n$. Then, in $B_\alpha \cup E_\beta$ there are at least n blocks; so, according to nR1, it follows that $T_\alpha \neq T_\beta$ provided that $B_\alpha \neq B_\beta$. Namely, if $T_\alpha = T_\beta$, then some n different blocks from $B_\alpha \cup B_\beta$ have two different common points - T and $T_\alpha = T_\beta$, but this is a contradiction with nR1.

So, the number of different common points of all the possible unordered (n-1)-tuples of blocks from the set $\{b_1,\dots,b_{i-1},b_{i+1},\dots,b_k\}$ with the block b_i equals the cardinality of the set

$$\{1,\dots,k-1\}^{(n-1)}$$

i.e., this is the number $\binom{k-1}{n-1}$.

This number is not greater than the number of points in b_i, i.e., not greater than q^{n-1} (statement 2). So, it holds:

$$\binom{k-1}{n-1} \le q^{n-1},$$

i.e., $(k-1)\cdot\ldots\cdot(k-n+1) \le (n-1)!\cdot q^{n-1}$.

REMARK. It can be found in [6] and [7] that

$$k \le (n-1)q + 1.$$

In [11] it is proved that

$$k \le n + q - 1 .$$

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REZIME

O (k,n,q) - MREŽAMA

(k,n)- rešetke, $n \in N\setminus\{1\}$, $k \in N\setminus\{1,\dots,n\}$, predstavljaju jednu generalizaciju k-rešetaka, $k \in N\setminus\{1,2\}$; (k,2)-rešetke su, naime, k-rešetke [8-9]. Konačne (k,n)-rešetke reda $q \in N\setminus\{1\}$ zovemo i (k,n,q)-rešetke [7]. U ovom radu se utvrdjuje jedna veza izmedju k,n i q.

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ON FUZZY QUOTIENT ALGEBRAS

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ABSTRACT

For an algebra A and a complete lattice L, one can consider a fuzzy congruence relation $\bar{\rho}$ (defined in [1]). Here we define a quotient algebra $A/\bar{\rho}$. Since every fuzzy congruence relation is a special union of a family of ordinary congruences on the same algebra, it is interesting to consider the relationship between $A/\bar{\rho}$ and the quotient algebra A/ρ by any of the congruences of the family. We prove that there is always a homomorphism from $A/\bar{\rho}$ to A/ρ, and we give the necessary and sufficent conditions for it to be an isomorphism. We also consider the fuzzy subalgebras (defined as in [2]) of A, and A/ρ, and assuming that these mappings preserve the homomorphism, we prove that a fuzzy subalgebra $\bar{A}$ of A induces $\bar{A}/\rho$ (of A/ρ) and vice versa. Using the homomorphism from $A/\bar{\rho}$ onto A/ρ, we finally determine the connection between the corresponding fuzzy subalgebras.

1. Let S be an unempty set, and $L = (L,\wedge,\vee,0,1)$ a complete lattice. A <u>fuzzy set</u> $\bar{S}$ on S (or, a fuzzy subset $\bar{S}$ of S) is any mapping $\bar{S}: S \to L$ ([3]).

If $\bar{S}_1$ and $\bar{S}_2$ are two fuzzy sets on S, then the relation $\subseteq$ and the operations $\cap$ and $\cup$ are defined as follows ([3]):

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Key words and phrases: Fuzzy sets.

$$\bar{S}_1 \subseteq \bar{S}_2 \quad \text{iff for all } x \in S \quad \bar{S}_1(x) \le \bar{S}_2(x) \ ;$$

$$\bar{S}_1 \cap \bar{S}_2 : S \to L, \quad \text{and} \quad (\bar{S}_1 \cap \bar{S}_2)(x) = \bar{S}_1(x) \wedge \bar{S}_2(x) \ ;$$

$$\bar{S}_1 \cup \bar{S}_2 : S \to L, \quad \text{and} \quad (\bar{S}_1 \cup \bar{S}_2)(x) = \bar{S}_1(x) \vee \bar{S}_2(x) \ ;$$

(The operations on the right are those from L).

It is well known ([3]) that for any fuzzy set $\bar{S}$ on S the following equality holds:

1.1. $\bar{S} = \bigcup_{p \in L} p \cdot (S_p)$, where $S_p \subseteq S$, $x \in S_p$ iff $\bar{S}(x) \ge p$,

and
$$(S_p) : S \to L, \ (S_p)(x) = \begin{cases} 1, & \text{if } x \in S_p \\ 0' & \text{if } x \notin S_p \end{cases}$$

(the characteristic function of S_p). Here also

$$(p \cdot (S_p))(x) = p \wedge (S_p)(x) \ .$$

(From now on, we shall identify S_p with its characteristic function (S_p)). Clearly, if $x \in S$,

$$\bar{S}(x) = \bigvee_{p \in L} p \wedge S_p(x) .$$

It is also known that from 1.1. it follows that for $p, q \in L$, $p \le q$ implies $S_q \subseteq S_p$.

Let $A = (S, F)$ be an algebra. A <u>fuzzy congruence relation</u> $\bar{\rho}$ on A is a fuzzy relation on S, i.e. ([1])

$$\bar{\rho} : S^2 \to L, \quad \text{such that}$$

(1) $(\forall x \in S)(\bar{\rho}(x,x) = 1)$;

(2) $(\forall x, y \in S)(\bar{\rho}(x,y) = \bar{\rho}(y,x))$;

(3) $(\forall x, y, z \in S)(\bar{\rho}(x,y) \ge \bigvee_{z \in S} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y)))$;

(4) If $\bar{\rho}(x_1, y_1) = p_1, \dots, \bar{\rho}(x_n, y_n) = p_n$, $f_n \in F(n) \subseteq F$ [1], then $\bar{\rho}(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \ge \bigwedge_{i=1}^{n} p_i$.

1.2. If $\bar{\rho}$ is a fuzzy congruence relation on A, then

$$\bar{\rho} = \bigcup_{p \in L} p \cdot \bar{\rho}_p \ ,$$

1) $F(n)$ is a set of n-ary operations from F.

where ρ_p are congruences on A, and $p \leq q$ implies $\rho_q \subseteq \rho_p$. Here, as in 1.1.,

$$\bar{\rho}(x,y) = \bigvee_{p \in L} p \wedge \rho_p(x,y), \rho_p(x,y) = \begin{cases} 1 , & \text{if } \bar{\rho}(x,y) \geq p \\ 0 , & \text{otherwise.} \end{cases} \quad ([1]).$$

The following definitions are similar to those in [2]:

A mapping $\bar{A}: S \to L$ is a <u>fuzzy subalgebra</u> of $A = (S,F)$, iff for every $f \in F(n)$, and for $x_1,\ldots,x_n \in S$

$$\bar{A}(f(x_1,\ldots,x_n)) \geq \bar{A}(x_1) \wedge \ldots \wedge \bar{A}(x_n) .$$

Let $A = (S,F)$ and $B = (T,F)$ be two algebras from the same similarity class. If f is a homomorphism from A to B, then f is said to be a <u>fuzzy homomorphism</u> from a fuzzy subalgebra $\bar{A}$ of A to the fuzzy subalgebra $\bar{B}$ of B iff

$$\bar{A} \subseteq B \circ f, \quad \text{i.e. iff for every } x \in S,\ \bar{A}(x) \leq \bar{B}(f(x)) .$$

2. Let $A = (S,F)$ be an algebra, $L = (L,\wedge,\vee,0,1)$ a complete lattice, and $\bar{\rho}$ a fuzzy congruence relation on A. For an $x \in S$, let

$$[x]_{\bar{\rho}} : S \to L, \quad [x]_{\bar{\rho}}(a) = \bar{\rho}(x,a), \text{ for all } a \in S .$$

Let us make the following definition:

$$S/\bar{\rho} = \{[x]_{\bar{\rho}} \ ; \ x \in S\} .$$

If ρ_p, $p \in L$, is one of the congruences from the family determined by $\bar{\rho}$ in 1.2., then let $[x]_{\rho_p}$ be a characteristic function for $|x|_{\rho_p} \in S/\rho_p$, i.e.

$$[x]_{\rho_p}(a) \stackrel{def}{=} \begin{cases} 1 , & \text{if } a \in |x|_{\rho_p} \\ 0 , & \text{if } a \notin |x|_{\rho_p} \end{cases} .$$

Then, $[x]_{\rho_p}(a) = \rho_p(x,a)$. Now we have:

2.1. $$[x]_{\bar{\rho}} = \bigcup_{p \in L} (p \cdot [x]_{\rho_p}),\ x \in S .$$

P r o o f. $[x]_{\bar{\rho}}(a) = \bar{\rho}(x,y) = (\bigcup_{p\in L} p\cdot\rho_p)(x,a) =$

$$= \bigvee_{p\in L} (p\cdot\rho_p)(x,a) = \bigvee_{p\in L} (p\cdot\rho_p(x,a)) =$$

$$= \bigvee_{p\in L} (p\cdot[x]_{\rho_p}(a)) = \bigvee_{p\in L} (p\cdot[x]_{\rho_p})(a) =$$

$$= (\bigcup_{p\in L} (p\cdot[x]_{\rho_p}))(a) .$$

For every $f \in F(n)$, define an operation $\bar{f}$ on $S/\bar{\rho}$: If $x_1,\dots,x_n \in S$

$$\bar{f}([x_1]_{\bar{\rho}},\dots,[x_n]_{\bar{\rho}}) \stackrel{\text{def}}{=} \bigcup_{p\in L} (p\cdot f([x_1]_{\rho_p},\dots,[x_n]_{\rho_p})),$$

where, as we noted, we use the characteristic function $[x]_{\rho_p}$ instead of a class $|x|_{\rho_p}$, and thus

$$f([x_1]_{\rho_p},\dots,[x_n]_{\rho_p}) = [f(x_1,\dots,x_n)]_{\rho_p} .$$

Now we can prove the following statement:

2.2. For all $x_1,\dots,x_n \in S$.

$$\bar{f}([x_1]_{\bar{\rho}},\dots,[x_n]_{\bar{\rho}}) = [f(x_1,\dots,x_n)]_{\bar{\rho}} .$$

Proof.

$$\bar{f}([x_1]_{\bar{\rho}},\dots,[x_n]_{\bar{\rho}}) = \bigcup_{p\in L} (p\cdot f([x_1]_{\rho_p},\dots,[x_n]_{\rho_p})) =$$

$$= \bigcup_{p\in L} (p\cdot[f(x_1,\dots,x_n)]_{\rho_p}) = [f(x_1,\dots,x_n)]_{\bar{\rho}} .$$

We can thus define a new algebra :

$$A/\bar{\rho} = (S/\bar{\rho},\bar{F}), \text{ where } F = \{\bar{f};\ f\in F\} .$$

The following two propositions deal with some properties of fuzzy equivalence relations, and they will be used in considering the connection between $A/\bar{\rho}$ and the usual factor algebra A/ρ_p .

2.3. Let $\bar{\rho}$ be a fuzzy equivalence relation on S. Let $a,b \in S$, $a \neq b$. Then for every $x \in S$,

$$\bar{\rho}(a,x) = \bar{\rho}(b,x) \quad \text{iff} \quad \bar{\rho}(a,b) = 1.$$

P r o o f. Let $\bar{\rho}(a,b) = 1$, $a \neq b$. Then, because of (3),

$\bar{\rho}(a,x) \geq \bar{\rho}(a,b) \wedge \bar{\rho}(b,x)$, and thus

(5) $\bar{\rho}(a,x) \geq \bar{\rho}(b,x)$.

Exactly in the same way, using (2) and (3), we get

(6) $\bar{\rho}(b,x) \geq \bar{\rho}(a,x)$.

From (5) and (6), it follows that $\bar{\rho}(a,x) = \bar{\rho}(b,x)$.

Let now $\bar{\rho}(a,x) = \bar{\rho}(b,x)$, $a \neq b$. Then for $x = a$

$1 = \bar{\rho}(a,a) = \bar{\rho}(b,a)$, i.e. $\bar{\rho}(a,b) = 1$.

2.4. Let $\bar{\rho}$ be a fuzzy equivalence relation on S, and $a,b \in S$. Then

$$[a]_{\bar{\rho}} = [b]_{\bar{\rho}} \quad \text{iff} \quad [a]_{\rho_1} = [b]_{\rho_1} .$$

P r o o f. Let $[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}}$. Suppose that there is an $x \in S$, such that $\bar{\rho}(a,x) = \bar{\rho}(b,x) = 1$. Using (3), we get that $\bar{\rho}(a,b) = 1$. But then for every $x \in S$, by 2.3., $\bar{\rho}(a,x) = \bar{\rho}(b,x)$, i.e. $[a]_{\bar{\rho}} = [b]_{\bar{\rho}}$, which is a contradiction.

If $[a]_{\bar{\rho}} = [b]_{\bar{\rho}}$, then for every $p \in L$, $[a]_{\rho_p} = [b]_{\rho_p}$, and hence $[a]_{\rho_1} = [b]_{\rho_1}$.

Consider now the above defined factor algebra $A/\bar{\rho} = (S/\bar{\rho}, F)$, for a given algebra A, by a fuzzy congruence relation $\bar{\rho}$, and a factor algebra $A/\rho_p = (S/\rho_p, F)$, where ρ_p, $p \in L$, is any of the congruences from the collection defined in 1.2.

2.5. The mapping $h: S/\bar{\rho} \to S/\rho_p$, defined with

$$h([a]_{\bar{\rho}}) = [a]_{\rho_p} , \quad a \in S ,$$

is a homomorphism from $A/\bar{\rho}$ onto A/ρ_p .

P r o o f. Since for every $a \in S$, and $[a]_{\rho_p} \in S/\rho_p$ there is an original in $S/\bar{\rho}$, namely $[a]_{\bar{\rho}}$, h is onto. We also have

$$h(\bar{f}([x_1]_{\bar{\rho}}, \dots, [x_n]_{\bar{\rho}})) = h([f(x_1,\dots,x_n)]_{\bar{\rho}}) =$$
$$= f([x_1]_{\rho_p}, \dots, [x_n]_{\rho_p}) = f(h([x_1]_{\bar{\rho}}),\dots,h([x_n]_{\bar{\rho}})) ,$$

proving that h is a homomorphism.

2.6. $A/\bar{\rho} \cong A/\rho_1$, $1 \in L$.

P r o o f. We have to prove that h (defined in 2.5.) is 1-1. By 2.4. we have

$$[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}} \quad \text{iff} \quad [a]\rho_1 \neq [b]\rho_1 ,$$

proving that h satisfies this property.

A fuzzy relation $\bar{\rho}: S^2 \to L$ is <u>strongly reflexive</u>, if the following is satisfied:

$$\bar{\rho}(x,y) = 1 \quad \text{iff} \quad x = y.$$

2.7. $A/\bar{\rho} \cong A$ iff $\bar{\rho}$ is a strongly reflexive fuzzy congruence relation on A.

P r o o f. If $A/\bar{\rho} \cong A$, then by 2.6. $A/\rho_1 = A$, i.e. ρ_1 is a diagonal.

On the other hand, if $\bar{\rho}$ is strongly reflexive, then ρ_1 is a daigonal, and thus $|a|_{\rho_1} = a$. Now, by 2.4., $a \neq b$ implies $[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}}$ i.e. the mapping h $(h([a]_{\bar{\rho}}) = a)$ is an isomorphism.

3. Consider now the algebras $A = (S,F)$, $A/\bar{\rho} = (S/\bar{\rho}, \bar{F})$, and for every $p \in L$, $A/\rho_p = (S/\rho_p, F)$, where $\bar{\rho} = \bigcup_{p \in L} p \cdot \rho_p$ is a fuzzy congruence relation on A, ρ_p $(p \in L)$ is an ordinary congruence relation (as in 1.2.), and $L = (L, \wedge, \vee, 0, 1)$ is a complete lattice. For each of these algebras one can define the corresponding fuzzy subalgebras, as in 1.3. Here we shall discuss the relationship between these fuzzy structures.

3.1. Let $\bar{A}:S \to L$ be a fuzzy subalgebra of A. Let also ρ be an ordinary congruence relation on A. Define the mapping $\bar{A}/\rho : S/\rho \to L$, so that $\bar{A}/\rho([x]_\rho) \overset{def}{=\!=} \bigvee_{y \in [x]_\rho} \bar{A}(y)$. Now, if L is distributive, then

a) $\bar{A}/\rho$ is a fuzzy subalgebra of A/ρ, and

b) $f_\rho = \ker \rho$ is a fuzzy homomorphism from A/ρ onto $\bar{A}/\rho$.

P r o o f. a) Let $f \in F(n)$, $x_1, \ldots, x_n \in S$. Then

$$\bar{A}/\rho(f([x_1]_\rho, \ldots, [x_n]_\rho)) = \bar{A}/\rho([f(x_1, \ldots, x_n)]_\rho) =$$

$$\bigvee(\bar{A}(y);\ y \in [f(x_1, \ldots, x_n)]_\rho) \geq$$

$$\bigvee(\bar{A}(f(y_1, \ldots, y_n);\ y_1 \in [x_1]_\rho, \ldots, y_n \in [x_n]_\rho\ ) \geq$$

$$\bigvee(\bar{A}(y_1) \wedge \ldots \wedge \bar{A}(y_n);\ y_1 \in [x_1]_\rho, \ldots, y_n \in [x_n]_\rho) =$$

$$\bigvee_{y_1 \in [x_1]_\rho}(\bar{A}(y_1) \wedge \ldots \wedge \bigvee_{y_n \in [x_n]_\rho} \bar{A}(y_n)\ ,$$

where we use the definition of a fuzzy subalgebra, and the fact that L is distributive.

b) Let $x \in S$, and $f\ = \ker \rho$. Then

$$\bar{A}/\rho(f_\rho(x)) = \bar{A}/\rho([x]_\rho) = \bigvee_{y \in [x]_\rho} \bar{A}(x) \geq \bar{A}(x)\ , \quad \text{i.e.}$$
$$\bar{A} \subseteq \bar{A}/\rho \circ f_\rho\ .$$

The following corollary shows that, in the family of all fuzzy subalgebras of A/ρ, the one defined in 3.1. (and provided that L is distributive) is the smallest one - as a fuzzy set - satisfying 3.1. (b).

3.2. Let $\bar{A}/\rho$ be as in 3.1. If $\bar{A}_1/\rho : S/\rho \to L$ is any fuzzy subalgebra of A/ρ satisfying 3.1. (b), then

$$\bar{A}/\rho \subseteq \bar{A}_1/\rho\ .$$

P r o o f. By the definition of fuzzy homomorphism, if $x \in S$, $\bar{A}_1/\rho([x]_\rho) \geq \bar{A}(x)$. But then for every $y \in [x]_\rho$,
$\bar{A}_1/\rho([y]_\rho) \geq \bar{A}(y)$, and hence

$$\bar{A}_1/\rho([x]_\rho) \geq \bigvee_{y \in [x]_\rho} \bar{A}(y) = \bar{A}/\rho([x]_\rho)\ .$$

If we start with a fuzzy subalgebra of A/ρ, then it induces a fuzzy subalgebra of A in the following way.

3.3. Let ρ be a congruence relation on A, and $\bar{A}/\rho : S/\rho \to L$ an arbitrary fuzzy subalgeba of A/ρ. If $\bar{A}$ is a mapping from S to L, such that for $x \in S$ $\bar{A}(x) \overset{\text{def}}{=\!=} \bar{A}/\rho([x]_\rho)$, then:

a) $\bar{A}$ is a fuzzy subalgebra of A,

and

b) $f_\rho = \ker\rho$ is a fuzzy homomorphism from $\bar{A}$ onto $\bar{A}/\rho$.

P r o o f. a) For $x_1,\dots,x_n \in S$, $f \in F(n)$,

$$\bar{A}(f(x_1,\dots,x_n)) = \bar{A}/\rho([f(x_1,\dots,x_n)]_\rho) =$$

$$\bar{A}/\rho(f([x_1]_\rho,\dots,[x_n]_\rho)) \geq$$

$$\bar{A}/\rho([x_1]_\rho) \wedge \dots \wedge \bar{A}/\rho([x_n]_\rho) = \bar{A}(x_1) \wedge \dots \wedge \bar{A}(x_n),$$

since $\bar{A}/\rho$ is a fuzzy subalgebra of A/ρ, by assumption.

b) For every $x \in S$,

$\bar{A}(x) \leq \bar{A}/\rho(f_\rho(x))$, since the equality holds by definition.

3.4. Let $\bar{A}$ be as in 3.3. If $\bar{A}_1 : S \to L$ is any fuzzy subalgebra of A satisfying 3.3. (b), then $\bar{A}_1 \subseteq A$.

P r o o f. By the definition of a fuzzy homomorphism, for any $\bar{A}_1$ and $x \in S$

$$\bar{A}_1(x) \leq \bar{A}/\rho([x]_\rho) = \bar{A}(x), \quad \text{i.e.} \quad \bar{A}_1 \subseteq \bar{A}.$$

As a direct consequence of the definition of fuzzy homomorphism, we have the following two lemmas.

3.5. If f is an isomorphism from algebra A to A_1, and f and f^{-1} are both fuzzy homomorphism relative to the subalgebras $\bar{A}$ and $\bar{A}_1$ of A and A_1 respectively, then for every $x \in S$ $\bar{A}(x) = \bar{A}_1(f(x))$.

3.6. If ρ and σ are two congruences on $A=(S,F)$ and $\rho\subseteq\sigma$, and if the homomorphism $h:S/\rho\to S/\sigma$ is also a fuzzy homomorphism from a fuzzy subalgebra $\bar{A}/\rho$ of A/ρ to the fuzzy subalgebra $\bar{B}/\sigma$ of A/σ, then for every $x\in S, \bar{A}/\rho([x]\rho)\leq\bar{B}/\rho([x]\sigma)$.

Consider now an arbitrary fuzzy congruence relation $\bar{\rho}=\bigcup_{p\in L} p\cdot\rho_p$ on $A=(S,F)$. In part 2 we have proved that there is a homomorphism from $A/\bar{\rho}$ to A/ρ_p, for every $p\in L$, and that $A/\bar{\rho}\cong A/\rho_1$. If we considered the fuzzy subalgebras of these structures, we get the following consequences of the preceding propositions.

3.7. Let $\bar{A}/\bar{\rho}$ be a fuzzy subalgebra of $A/\bar{\rho}$, and for every $p\in L$, take a fuzzy subalgebra $\bar{A}/\rho_p$ of A/ρ_p, such that the homomorphism $h:A/\bar{\rho}\to A/\rho_p$ is preserved under the formation of fuzzy subalgebras (i.e. h is the corresponding fuzzy homomorphism), then for every $x\in S$, $\bar{A}/\bar{\rho}([x]_{\bar{\rho}})\leq\bar{A}/\rho_p([x]_{\rho_p})$, and $\bar{A}/\bar{\rho}([x]_{\bar{\rho}})=\bar{A}/\rho_1([x]\rho_1)$.

P r o o f. By 2.5., 2.6., 3.5., and 3.6. .

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REZIME

O RASPLINUTIM FAKTOR ALGEBRAMA

U vezi sa algebrom A i kompletnom mrežom L, posmatraju se rasplinute kongruencije (definisane u [1]). Uvodi se pojam faktor algebre po rasplinutoj kongruenciji: $A/\bar{\rho}$. Kako je svaka rasplinuta kongruencija posebna unija familije običnih kongruencija na istoj algebri, od interesa je posmatrati odnos izmedju $A/\bar{\rho}$ i A/ρ, za proizvoljnu kongruenciju ρ familije. Dokazano je da uvek postoji homomorfizam izmedju $A/\bar{\rho}$ i A/ρ i dati su potrebni i dovoljni uslovi za koje je to izomorfizam. Razmatraju se i rasplinute podalgebre (definisane slično kao u [2]) algebri A i A/ρ i pretpostavljajući da ta preslikavanja očuvavaju homomorfizam, dati su uslovi pod kojima jedno od njih indukuje drugo i obrnuto. Takodje je opisana veza izmedju rasplinutih podalgebri od $A/\bar{\rho}$ i A/ρ.

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SEMIGROUPS WHOSE PROPER IDEALS ARE ARCHIMEDEAN SEMIGROUPS

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ABSTRACT

In this paper semigroups whose proper (left) ideals are archimedean (left archimedean, t-archimedean, power joined) semigroups are considered.

In [10] T.E.Nordahl studied commutative Q-semigroups, i.e. commutative semigroups in which every proper ideal is power joined. C.S.H.Nagore, [8] extended Nordahl's results to quasi-commutative semigroups. A.Cherubini and A.Varisco in [6] considered Putcha's Q-semigroups. Weakly commutative semigroups in which every proper right ideal is power joined are studied by the author in [1]. B.Pondělíček, [12] considered uniform semigroups whose proper quasi-ideals are power joined. A characterization of Q-semigroups in the general case is given by A.Nagy, [9].

In the present paper we shall describe semigroups in which every proper two-sided ideal is an archimedean semigroup, (Theorem 1.) and in this way a generalization of the previous results is given. Theorem 1. is also a generalization of some results of [2,3,5]. Also, we shall decribe semigroups in which

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every proper left ideal is an archimedean (left archimedean, t-archimedean, power joined) semigroup, (Theorems 2,3,4,5.). At the end we describe semigroups in which every proper subsemigroup is power joined, (Theorem 6.).

Throughout this paper let N denote the set of all positive integers.

A semigroup S is archimedean if for any $a,b \in S$ there exists $n \in N$ for which $a^n \in SbS$, [11]. S is left archimedean if for every $a,b \in S$ there exists $n \in N$ such that $a^n \in Sb$, [13] (see also [14]). S is t-archimedean if for every $a,b \in S$ there exists $n \in N$ for which $a^n \in bS \cap Sb$, [13]. S is power joined if for every $a,b \in S$ there exist $m,n \in N$ such that $a^n = b^n$, [11]. S is special power joined if for every $a,b \in S$ there is an $n \in N$ such that $a^n = b^n$, [4].

Underfined notions and terminology are as in [7] and [11].

Let I(S)(L(S)) denote the union of all proper two-sided (left) ideals of a semigroup S.

THEOREM 1. *Every proper two-sided ideal of* S *is an archimedean subsemigroup of* S *if and only if* I(S) *is an archimedean subsemigroup of* S.

P r o o f. If all proper two-sided ideals of S are archimedean and $a,b \in I(S)$, then there is a proper two-sided ideal I of S with $a, aba \in I$ and there exists $n \in N$ such that

$$a^n \in IabaI \subseteq I(S)bI(S).$$

Thus I(S) is archimedean.

Conversely, let I(S) be archimedean and I be a proper two-sided ideal of S. Then for $a,b \in I$ there is an $n \in N$ such that $a^n = xby$ for some $x,y \in I(S)$. Hence $a^{n+2} = axbya$, where $ax, ya \in I$, and therefore I is archimedean.

LEMMA 1. *Let* L *be a (proper) left ideal of* S . *Then* L *is maximal if and only if*

(i) $S \setminus L = \{a\}$, $a^2 \in L$

or

(ii) $S \setminus L \subseteq Sa$ *for every* $a \in S \setminus L$.

P r o o f. If L is a maximal left ideal of S, then we have the two cases: (i) There is an $a \in S \setminus L$ such that $Sa \subseteq L$. In this case $L \cup \{a\} = S$. Hence $S \setminus L = \{a\}$, $a^2 \in L$. (ii) For every $a \in S \setminus L$, $Sa \not\subseteq L$. Then $L \cup Sa = S$. Hence, $S \setminus L \subseteq Sa$ for every $a \in S \setminus L$.

The converse is obvious.

LEMMA 2. *Let* L(S) *be as in the case (ii) od Lemma 1. then*

$$S \setminus L(S) = \{x \in S : S = Sx\}$$

ia s subsemigroup of S.

P r o o f. For $a \in S \setminus L(S)$ we have that $S = L(S) \cup (S \setminus L(S)) = a \cup Sa$, so $L(S) \subseteq Sa$. From this and $S \setminus L(S) \subseteq Sa$ we have that $S = Sa$ for every $a \in S \setminus L(S)$. Conversely, let $S = Sa$ for every $a \in S \setminus L(S)$, then $S \setminus L(S) \subseteq Sa$, $a \in S \setminus L(S)$. Therefore $S \setminus L(S) = \{x \in S : S = Sx\}$ and it is clear that $S \setminus L(S)$ is a subsemigroup of S.

LEMMA 3. *Every left ideal of an archimedean (left archimedean,* t-*archimedean, power joined, special power joined) semigroup* S *is an archimedean (left archimedean,* t-*archimedean, power joined, special power joined) subsemigroup of* S.

P r o o f. Let L be an arbitrary left ideal of an archimedean semigroup S and $a, b \in L$. Then $a^n = xb^2y$ for some $x, y \in S$ and $n \in N$. It follows from this that $a^{n+1} = xbbya$ and $xb, ya \in L$.

THEOREM 2. *The following conditions are equivalent on a semigroup* S*:*

(1) *Every proper left ideal of* S *is archimedean;*

(2) L(S) *is archimedean;*

(3) S *satisfies one of the following conditions:*

(i) S *is archimedean;*

(ii) S *has a maximal left ideal* M *which is an archimedean semigroup and* $M \subseteq Ma$ *for any* $a \in S \setminus M$.

P r o o f. (1) => (2). If S is left simple, then S is archimedean. Assume that S is not left simple. If $a,b \in L(S)$, then there is a proper left ideal L of S such that $a, ba \in L$. Hence,

$$a^n \in LbaL \subseteq L(S)bL(S)$$

for some $n \in N$ and therefore L(S) is archimedean.

(2) => (3). If $L(S) \neq S$, then $M = L(S)$ is a maximal left ideal of S and by Lemma 1. we have that $S \setminus M = \{a\}$, $a^2 \in M$ or $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$. If $S \setminus M = \{a\}$, $a^2 \in M$, then S is archimedean. If $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$, then by Lemma 2. $T = S \setminus M$ is a subsemigroup of S. From $Sa = S$ $(a \in T)$ we have that $S = Ma \cup Ta \subseteq Ma \cup T \subseteq S$, i.e. $S = Ma \cup T$. Hence, $M \subseteq Ma$ for every $a \in S \setminus M$.

(3) => (1). If (i) holds, then by Lemma 3. every left ideal of S is archimedean. Let (ii) holds and let L be a proper left ideal of S. If $L \subseteq M$, then L is archimedean, (Lemma 3.). If $L \not\subseteq M$, then $L \cap (S \setminus M) \neq \emptyset$. For $a \in L \cap (S \setminus M)$ we have $M \subseteq Ma \subseteq L$, which is not possible.

THEOREM 3. *Every proper left ideal of a semigroup* S *is a left archimedean subsemigroup of* S *if and only if one the following conditions hold:*

1^o S *is left archimedean;*

2^o S *contains exactly two left ideals* L_1 *and* L_2 *which are left simple semigroup and* $S = L_1 \cup L_2$ *;*

3^o S *has a maximal left ideal* M *which is left archimedean and* $M \subseteq Ma$ *for every* $a \in S \setminus M$.

P r o o f. Let all proper left ideals of S are left archimedean. If $L(S) \neq S$, then $M = L(S)$ is a maximal left ideal of S which is left archimedean. By Lemma 1. we have $S \setminus M = \{a\}$, $a^2 \in M$ or $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$. In the last case we have by Theorem 2. and Lemma 2. that $M \subseteq Ma$ for every $a \in S \setminus M$.

If $L(S) = S$ and for any two proper left ideals L_1, L_2 of S we have $L_1 \cap L_2 \neq \emptyset$, then S is left archimedean. Otherwise, there are left ideals L_1, L_2 of S with $L_1 \cap L_2 \neq \emptyset$. In this case $L_1 \cup L_2 = S$, since $L_1 \cup L_2$ is not left archimedean. Moreover, L_1 and L_2 are left simple semigroups and there exists no other proper left ideal L of S than L_1 and L_2. Consequently, if every proper left ideal of S is left archimedean, then we have one of the conditions $1^\circ, 2^\circ$ or 3°.

The converse follows immediately.

LEMMA 4. [1] *S is t-archimedean and left simple if and only if S is a group.*

THEOREM 4. *Let S not be left simple. Then every proper left ideal of S is t-archimedean if and only if one of the following conditions hold:*

1° S *is t-archimedean;*

2° S *contains exactly two left ideals* G_1, G_2 *which are groups and* $S = G_1 \cup G_2$;

3° S *has a maximal left ideal* M *which is a t-archimedean semigroup and* $M \subseteq Ma$ *for any* $a \in S \setminus M$.

P r o o f. Let every proper left ideal of S be t-archimedean. Then by Theorem 3. and Lemma 4. we have 2° or 3° or S is left archimedean. Assume that S is left archimedean. If $L(S) \neq S$, then $L(S)$ is a maximal left ideal of S and it is t-archimedean. By Lemmas 1. and 2. we have that $S \setminus L(S) = \{a\}$, $a^2 \in L(S)$ or $S \setminus L(S)$ is a subsemigroup of S. The last case is not possible and in the first case S is t-archimedean. If $L(S) = S$, then we can prove that S is of the type 1°.

The converse follows immediately.

The following theorem will be given without proof.

THEOREM 5. *Let* S *not be left simple. Then every proper left ideal of* S *is power joined if and only if one of the following conditions hold:*

1^o S *is power joined;*

2^o S *contains exactly two left ideals* G_1, G_2 *which are periodic groups and* $S = G_1 \cup G_2$;

3^o S *has a maximal left ideal* M *which is power joined and* $M \subseteq Ma$ *for any* $a \in S \setminus M$.

THEOREM 6. *Every proper subsemigroup of* S *is power joined if and only* $|S| = 2$ *or* S *is power joined.*

P r o o f. Let S be not left simple. If any proper subsemigroup of S is power joined, then also any proper left ideal of S is power joined. Hence, by Theorem 5. we have one of the cases $1^o, 2^o$ or 3^o of this theorem. But, the cases 2^o and 3^o are possible only if $|S| = 2$. Indeed, let $S = G_1 \cup G_2$, where and G_2 are the disjoint left ideals of S which are periodic groups. If e and f are the units of G_1 and G_2, respectively, then it is clear that $S = \langle e,f \rangle$. Moreover, $ef \in G_2$ and $fe \in G_1$ and there exist $m,n \in N$ such that $f = (ef)^m$ and $e = (fe)^n$, so $ef = f$, $fe = e$. Therefore $S = \langle e,f \rangle = \{e,f\}$, i.e. $|S| = 2$. If we have 3^o, then $M = L(S)$, so $S \setminus M = \{a \in S : Sa = S\}$ is a subsemigroup of S (Lemma 2.). From $Sa = S$ $(a \in T = S \setminus M)$ we have $S = Ma \cup Ta$. If $Ma = M$, then we have $Ta = T$. Assume that T is not left simple. Then there is an element $a \in T$ with $Ta \nsubseteq T$. But, in this case $M \nsubseteq Ma$, hence, $Ma = S$. Let $a = xa$ for some $x \in M$. Then

$$(ax)^2 = a(xa)x = a^2x, \dots, (ax)^n = a^nx \in M \quad (n \in N)$$

and thus

$$\{ax, a^2x, \dots, a^nx, \dots\} \cup \{a, a^2, \dots, a^n, \dots\}$$

is a subsemigroup of S. Since this subsemigroup is not power joined it is equal to S and thus

$$M = \{ax, a^2x, \dots\} \quad , \quad T = \{a, a^2, \dots\} \ .$$

Consequently, $x = a^k x$ for some $k \in N$. But then

$$a = xa = a^k(xa) = a^{k+1}$$

and T is a group. This is a contradiction. Therefore, T is a left simple semigroup and by Lemma 4. T is a subgroup of S. Let e be the unity of T. Then by 3° we have $M \subseteq Me$ and thus for any $x \in M$ there is some $y \in M$ with $x = ye$. Hence, $xe = ye^2 = ye = x$. For such an element $x \in M$ we have $(ex)^n = ex^n \in M$ $(n \in N)$ and $\{ex^2, ex^3, \dots\} \cup \{e\}$ is a subsemigroup of S. This subsemigroup is not power joined, and thus it is equal to S. Consequently, $M = \{ex^2, ex^3, \dots\}$, $T = \{e\}$. But in this case $ex = ex^k = (ex)^k$ for some $k > 1$ and $M = \{ex, ex^2, \dots\}$ is a group. For the unity $(ex)^{k-1} = ex^{k-1}$ of this group we have the subsemigroup $\{ex^{k-1}, e\}$ of S which is not power joined. Hence, $S = \{ex^{k-1}, e\}$, i.e. $|S| = 2$.

Now, let S be left simple. Then we have two cases: (i) S is right simple. In this case S is a periodic group. (ii) If S is not right simple, then using the dual of Theorem 5. we have, as in the case that S is not left simple, that S is power joined or $|S| = 2$.

The converse is obvious.

COROLLARY 1. [3] *Every proper subsemigroup of a semigroup* S *is special power joined if and only if* $|S| = 2$ *or* S *is special power joined.*

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REZIME

PODGRUPE U KOJIMA SU SVI PRAVI IDEALI ARHIMEDOVSKE POLUGRUPE

U ovom radu razmatraju se polugrupe u kojima su svi pravi (levi) ideali arhimedovske (levo arhimedovske, t-arhimedovske, stepeno vezane) polugrupe.

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ON BIPARTITE SCORE SETS

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ABSTRACT

A necessary and sufficient condition for sets of non-negative integers $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, $0 \le b_1 < b_2 < \ldots < b_n$ to be the score sets of a bipartite tournament is given.

A bipartite tournament is a complete asymmetric bipartite digraph. The number of edges oriented from a vertex is called a score. Two sequences $a_1 \le a_2 \le \ldots \le a_k$ and $b_1 \le b_2 \le \ldots \le \ldots \le b_\ell$, corresponding to the scores of the bipartite sets of a bipartite tournament, are called a score sequence. The sets $A = \{a_i | 1 \le i \le k\}$ and $B = \{b_i | 1 \le i \le \ell\}$ of elements of the score sequences, are called score sets.

Throughout the paper we shall denote by $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ sets of non-negative integers such that $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_n$ where a_1 and b_1 are not both zero.

At the Fourth International Conference on Theory and Applications of Graphs in Kalamazoo 1981, K.B.Reid raised

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the problem of determining the score sets of bipartite tournaments. K.Wayland [4] found a necessary and sufficient condition for the existence of a bipartite tournament with bipartition (X,Y) and the score sets A and B, if $|X| > b_n$.

Since some bipartite tournaments exist only for $|X| = b_n$, it is very unlikely that a sensible necessary sufficient condition can be given for general case.

We present, using a constructive method, a necessary and sufficient condition for the existence of a bipartite tournament with the score sets $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

THEOREM. *The sets of non-negative integers* $A = \{a\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$ *are the score sets of some bipartite tournament if and only if one of the following conditions is satisfied:*

(a) $b_1 + b_2 + \ldots + b_n = (n-a-1)b_n$;

(b) $b_1 + b_2 + \ldots + b_n > (n-a-1)(b_n+1)$;

(c) $b_1 + b_2 + \ldots + b_n = (n-a-1)b_n + d$, $1 \leq d \leq n-a-1$

and there exist positive integers $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ *such that*

$$ab_n = \gamma_1(b_n - b_1) + \gamma_2(b_n - b_2) + \ldots + \gamma_{n-1}(b_n - b_{n-1}) .$$

P r o o f. Necessity. Firstly we prove the inequality

$$b_1 + b_2 + \ldots + b_{n-1} \geq (n-a-1)b_n . \tag{1}$$

Let T be a bipartite tournament whose bipartite sets X and Y have the score sequences

$$\underbrace{(a, a, \ldots, a)}_{\alpha} \text{ and } (\underbrace{b_1, \ldots, b_1}_{\beta_1}, \underbrace{b_2, \ldots, b_2}_{\beta_2}, \ldots, \underbrace{b_n, \ldots, b_n}_{\beta_n})$$

respectively, where $\alpha \geq 1$ and $\beta_i \geq 1$, $i=1,\ldots,n$. Denote by y_{i1}, $y_{i2}, \ldots, y_{i\beta_i}$, $i=1,\ldots,n$ vertices in Y having the score b_i and

by $X_{i1}, X_{i2}, \ldots, X_{i\beta_i}$ their insets, i.e. $X_{ij} = \{x \mid x \to y_{ij}\}$. Since every vertex of X has a score a, the sets X_{ij}, i=1,...,n, j=1,...,β_i are a covering of X such that every vertex of X is covered by precisely a insets. Thus,

$$|X_{i1}| = |X_{i2}| = \ldots = |X_{i\beta_i}| = |X| - b_i = \alpha - b_i, \quad i=1,\ldots,n$$

and

$$a\alpha = \sum_{i=1}^{n} \sum_{j=1}^{\beta_i} |X_{ij}| = \sum_{i=1}^{n} \beta_i(\alpha - b_i)$$

hold and we have

$$(2) \qquad \alpha = \frac{\beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_n b_n}{\beta_1 + \beta_2 + \ldots + \beta_n - a} .$$

As $\alpha \geq b_n$, we get from (2)

$$(3) \qquad \beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_{n-1} b_{n-1} \geq (\beta_1 + \beta_2 + \ldots + \beta_{n-1} - a) b_n .$$

From $\beta_i \geq 1$, i=1,...,n-1 and $0 \leq b_1 < b_2 < \ldots < b_n$ (1) follows.

Now we prove that the equality

$$(4) \qquad b_1 + b_2 + \ldots + b_{n-1} = (n-a-1) b_n + d, \quad 1 \leq d \leq n-a-1$$

implies $ab_n = \gamma_1(b_n - b_1) + \gamma_2(b_n - b_2) + \ldots + \gamma_{n-1}(b_n - b_{n-1})$ for some positive integers γ_i, i=1,...,n-1.

Let $y_1, y_2, \ldots, y_n$ be vertices in Y with the scores b_1, $b_2, \ldots, b_n$ and $X_1, X_2, \ldots, X_n$ their insets. Denote by s_o the total number of vertices in all other insets and by s_i the cardinality of X_i, i=1,...,n. Then the equalities

$$b_i = \alpha - s_i, \quad i=1,\ldots,n$$

$$s_1 + s_2 + \ldots + s_n + s_o = a\alpha$$

hold and imply

$$(5) \qquad b_i = (s_1 + \ldots + s_{i-1} + (1-a) s_i + s_{i+1} + \ldots + s_n + s_o)/a, \quad i=1,\ldots,n.$$

Substituting (5) in (4) we get

$$(n-a)s_n + s_o = d$$

and, since $s_n \geq 0$ and $1 \leq d \leq n-a-1$ (in particular $a < n$), we get

$$s_n = 0$$

i.e.

$$b_n = \alpha .$$

Hence (3) becomes an equality and hence

$$ab_n = \beta_1(b_n-b_1) + \beta_2(b_n-b_2) + \ldots + \beta_{n-1}(b_n-b_{n-1})$$

where $\beta_i \geq 1$, $i=1,\ldots,n-1$.

Setting $\gamma_i = \beta_i$, $i=1,\ldots,n$, we prove the statement.

Sufficiency. The structure of a bipartite tournament with the given bipartite sets X and Y is determined by the outsets of all the vertices of Y. We shall just construct these outsets.

According to the theorem we have to consider three cases.

(a) $b_1+b_2+\ldots+b_{n-1} = (n-a-1)b_n$.

Let T be the bipartite tournament with the bipartite sets $X = \{1,2,\ldots,b_n\}$ and $Y = \{y_1,y_2,\ldots,y_n\}$ and the dominance structure

$$y_i \to \{b_1+b_2+\ldots+b_{i-1}+1,\ldots,b_1+b_2+\ldots+b_{i-1}+b_i\},\ i=1,\ldots,n.$$

Summarizing is modulo b_n. Clearly the score of y_i, $i=1,\ldots,n$ is $s(y_i) = b_i$. From the construction and the fact that

$$\sum_{i=1}^{n} s(y_i) = \sum_{i=1}^{n} b_i = (n-a)b_n ,$$

it follows that every vertex of X is dominated by precisely (n-a) vertices of Y. As $|Y| = n$, the scores of all the vertices of X are a, and T is required tournament.

(b) $b_1+b_2+\ldots+b_{n-1} = (n-a-1)(b_n+1) + d, \quad d \geqslant 1$.

Set $X = \{1,2,\ldots,b_n+1\}$ and $Y = \{y_1,y_2,\ldots,y_{n-1},z_1,z_2,\ldots,z_d\}$, and construct the bipartite tournament T with the bipartite sets X and Y as follows:

$$y_i \to \{b_1+b_2+\ldots+b_{i-1}+1,\ldots,b_1+b_2+\ldots+b_{i-1}+b_i\}$$

for $i=1,\ldots,n-1$ and

$$z_j \to \{b_1+\ldots+b_{n-1}+(j-1)b_n+1,\ldots,b_1+\ldots+b_{n-1}+jb_n\}$$

for $j=1,\ldots,d$. Summarizing is modulo b_n+1.

Now $s(y_i) = b_i$, $i=1,\ldots,n-1$ and $s(z_j) = b_n$ $j=1,\ldots,d$. The equality

$$\sum_{i=1}^{n-1} s(y_i) + \sum_{j=1}^{d} s(z_j) = \sum_{i=1}^{n-1} b_i + d\,b_n = (n-a-1+d)(b_n+1)$$

implies that every vertex x of X is dominated by exactly $n-a-1+d$ vertices of Y and, therefore, has the score

$$\begin{aligned} s(x) &= |Y| - (n-a-1+d) \\ &= (n-1+d) - (n-a-1+d) = a \end{aligned}$$

This proves the construction.

(c) $b_1+b_2+\ldots+b_{n-1} = (n-a-1)b_n+d, \quad 1 \leq d \leq n-a-1$

and

$$ab_n = \gamma_1(b_n-b_1) + \gamma_2(b_n-b_2)+\ldots+\gamma_{n-1}(b_n-b_{n-1}), \ \gamma_i \geq 1 ,$$
$i=1,\ldots,n-1$.

In this case let $X = \{1,2,\ldots,b_n\}$,

$$Y = \{y_n,\ldots,y_{1\gamma_1},y_{21},\ldots,y_{2\gamma_2},\ldots,y_{n-1,1},\ldots,y_{n-1,\gamma_{n-1}},y_n\}$$

and

$$y_{ij} \to \{\gamma_1 b_1+\ldots+\gamma_{i-1}b_{i-1}+(j-1)b_i+1,\ldots,\gamma_1 b_1+\ldots+\gamma_{i-1}b_{i-1}+jb_i\}$$

for $i=1,\ldots,n-1$, $j=1,\ldots,\gamma_i$

$$y_n \to \{\gamma_1 b_1+\ldots+\gamma_{n-1}b_{n-1}+1,\ldots,\gamma_1 b_1+\ldots+\gamma_{n-1}b_{n-1}+b_n\}$$

Summarizing is modulo b_n. Similarly as, in (a) and (b), we obtain that $s(y_{ij}) = b_i$, $i=1,\ldots,n-1$, $s(y_n) = b_n$ and $s(x) = a$ for every $x \in X$.

This proves the theorem.

CORROLARY. *(Wayland [4]). Any finite nonempty set of non-negative integers, except {0}, may be the union of the score sets of some bipartite tournament.*

P r o o f. Let $a_1,a_2,\ldots,a_n$ be a set of nonnegative inregers such that $0 \leq a_1 < a_2 < \ldots < a_n$. Set $A = a_n$, and $B = \{a_1, a_2,\ldots,a_{n-1}\}$. Since $a_i \geq i-1$, $i=1,\ldots,n$, particularly $a_n \geq n-1$, the following inequality

$$a_1+a_2+\ldots+a_{n-2} > ((n-1)-a_n-1)(a_{n-1}+1)$$

holds.

According to the case (b), there exists a bipartite tournament with bipartite sets X and Y whose score sets are $\{a_n\}$ and $\{a_1,a_2,\ldots,a_{n-1}\}$, respectively.

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REZIME

O SKUPOVIMA SKOROVA BIPARTITNOG TURNIRA

U ovom radu daje se potreban i dovoljan uslov za skupove nenegativnih celih brojeva, $A=\{a\}$ i $B=\{b_1, b_2, \ldots, b_n\}$ $0 \le b_1 < b_2 < \ldots < b_n$, da budu skupovi skorova nekog bipartitnog turnira.

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SOME COMBINATORIAL IDENTITIES INSPIRED BY THE LAW OF ENERGY

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ABSTRACT

The investigations in paper [1] and the law of energy resulted combinatorials identities (1), (4) and (5) the proofs of which were indispensable for the physical interpretations of qualitative and quantitative points of the above mentioned paper. In this paper the identities (1), (4) and (5) are proved.

THEOREM 1.

$$(1)\quad \sum_{i=0}^{s} (-1)^i \binom{n}{s-i}\binom{p-s+i}{i} = \binom{n-p+s-1}{s} \quad \textit{for each}$$

$p,s,n \in N \cup \{0\}$ *and* $0 \le s \le p < n$.

P r o o f. Denote the functions on the left and right side of identity (1) as follows

$$F(p,s,n) = \sum_{i=0}^{s} (-1)^i \binom{n}{s-i}\binom{p-s+i}{i} \text{ and } G(p,s,n) = \binom{n-p+s-1}{s}.$$

The proofs are given by induction according to p. The theorem is obviously true for p=0 and p=1 and for each s and n, $0 \le s \le p < n$. Suppose that $F(p,s,n) = G(p,s,n)$ is true for some p and each s and n, $0 \le s \le p < n$. Now, we shall prove that $F(p+1,s,n) = G(p+1,s,n)$ for each s and n, $0 \le s \le p+1 < n$. First

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we shall prove that $F(p+1,s,n)=G(p+1,s,n)$ for each s and n, $0\le s\le p<n$ and then for $s=p+1$. Let us prove that

(2) $F(p+1,s,n)=F(p,s,n)-F(p,s-1,n)$, for each $0\le s\le p<n$.

Proof of (2)

$$F(p,s,n)-F(p,s-1,n)=\sum_{i=0}^{s}(-1)^i\binom{n}{s-i}\binom{p-s+i}{i}-\sum_{i=0}^{s-1}(-1)^i\binom{n}{s-1-i}\binom{p-s+1+i}{i}=$$

$$=\binom{n}{s}\binom{p-s}{0}+\sum_{i=1}^{s}(-1)^i\binom{n}{s-i}\binom{p-s+i}{i}-\sum_{i=0}^{s-1}(-1)^i\binom{n}{s-1-i}\binom{p-s+1+i}{i}=$$

$$=\binom{n}{s}\binom{p-s}{0}+\sum_{i=0}^{s-1}(-1)^{i+1}\binom{n}{s-1-i}\binom{p-s+i+1}{i+1}+$$

$$+\sum_{i=0}^{s-1}(-1)^{i+1}\binom{n}{s-1-i}\binom{p-s+i+1}{i}=\binom{n}{s}\binom{p-s}{0}+$$

$$+\sum_{i=0}^{s-1}(-1)^{i+1}\binom{n}{s-1-i}\binom{p+1-s+i+1}{i+1}=\binom{n}{s}\binom{p+1-s}{0}+$$

$$+\sum_{i=1}^{s}(-1)^i\binom{n}{s-i}\binom{p+1-s+i}{i}=$$

$$=\sum_{i=0}^{s}(-1)^i\binom{n}{s-i}\binom{p+1-s+i}{i}=F(p+1,s,n)\ .$$

We also need to prove that

(3) $G(p+1,s,n)=G(p,s,n)-G(p,s-1,n)$, for each $0\le s\le p<n$.

Proof of (3)

$$G(p,s,n)-G(p,s-1,n)=\binom{n-p+s-1}{s}-\binom{n-p+s-2}{s-1}=$$

$$=\binom{n-(p+1)+s-1}{s}=G(p+1,s,n)\ .$$

The equality $F(p+1,s,n)=G(p+1,s,n)$ for each $0\le s\le p<n$, follows from (2) and (3), bearing in mind that $F(p,s,n)=G(p,s,n)$ and $F(p,s-1,n)=G(p,s-1,n)$ for each $0\le s\le p<n$.

The equality $F(p+1,p+1,n) = G(p+1,p+1,n)$, for each $p+1 < n$ is the consequence of the following consideration:

$$\sum_{i=0}^{p+1}(-1)^i\binom{n}{p+1-i} = \sum_{i=0}^{p}(-1)^i\left(\binom{n-1}{p+1-i}+\binom{n-1}{p-i}\right) +$$

$$+ (-1)^{p+1}\binom{n}{0} = \binom{n-1}{p+1}+\binom{n-1}{p}-\binom{n-1}{p}-\binom{n-1}{p-1}+\binom{n-1}{p-1}+\ldots$$

$$+\ldots+(-1)^p\binom{n-1}{0}+(-1)^{p+1}\binom{n}{0}=\binom{n-1}{p+1}$$

THEOREM 2.

$$(4)\quad \sum_{s=0}^{p}\binom{n}{p-s}(1-x)^s x^{p-s} = \sum_{s=0}^{p}\binom{n-p+s-1}{s}x^s,$$

for each $p, s, n \in N \cup \{0\}$, $0 \le s \le p < n$ *and* $x \in R$.

P r o o f. Let us multiply (1) by x^s and perform the summation from $s = 0$ to $s = p$. Hence

$$\sum_{s=0}^{p}\sum_{i=0}^{s}(-1)^i\binom{n}{s-i}\binom{p-s-i}{i}x^s = \sum_{s=0}^{p}\binom{n-p+s-1}{s}x^s .$$

By substituting $s = k+i$ we change the order of summation on the left side L where it is obvious that the s run from s=k to s=p and i from i=0 to i=p-k, for fixed k. Hence

$$L = \sum_{k=0}^{p}\sum_{i=o}^{p-k}(-1)^i\binom{n}{k}\binom{p-k}{i}x^{k+i} = \sum_{k=0}^{p}\binom{n}{k}x^k\sum_{i=0}^{p-k}(-1)^i\binom{p-k}{i}x^i =$$

$$= \sum_{k=0}^{p}\binom{n}{k}x^k(1-x)^{p-k}$$

further, taking $p-k = s$, we have:

$$L = \sum_{s=0}^{p}\binom{n}{p-s}x^{p-s}(1-x)^s$$

the theorem is proved.

THEOREM 3.

$$(5)\quad \sum_{j=0}^{r-1} \binom{r-1}{j} (1-x)^j x^{r-1-j} \sum_{k=0}^{j} \frac{y^k}{k!} = 1 + \sum_{j=0}^{r-2} \sum_{k=0}^{j} \binom{j}{k} (1-x)^{k+1} x^{j-k} \frac{y^{k+1}}{(k+1)!},$$

for each $x,y \in R$ *and* $r \in N\setminus\{1\}$.

P r o o f. If in (4) we put $p=r-k-2$ and $n=r-1$, then $j-k-1=s$ and $j-k=s$ change, introduced for the left and right side of the identity respectively, the following result is obtained:

$$(6)\quad \sum_{j=k+1}^{r-1} (1-x)^{j-k-1} x^{k-1-j} = \sum_{j=k}^{r-2} \binom{j}{k} x^{j-k}$$

because $p-s=r-k-2-s=r-1-k-1-s=r-1-j$, $\binom{n}{p-s}=\binom{r-1}{r-1-j}=$

$$=\binom{r-1}{j},\ \binom{n-p+s-1}{s}=\binom{r-1-r+k+2+s}{s}=\binom{k+s}{s}=\binom{j}{s}=\binom{j}{j-s}=\binom{j}{k}.$$

The identity (6) is multiplied by

$$\frac{(1-x)^{k+1} y^{k+1}}{(k+1)!}$$

and the summation from $k=0$ to $k=r-2$ is performed, one is added to the left and right side, and the following equality is obtained:

$$1 + \sum_{k=0}^{r-2} \frac{y^{k+1}}{(k+1)!} \sum_{j=k+1}^{r-1} \binom{r-1}{j} (1-x)^j x^{r-1-j} = 1 + \sum_{k=0}^{r-2} \frac{y^{k+1}}{(k+1)!} \sum_{j=k}^{r-2} \binom{j}{k} x^{j-k} (1-x)^{k+1}.$$

Taking $k-1$ instead of k, we obtain:

$$(7)\quad \sum_{k=0}^{r-1}\sum_{j=k}^{r-1}\binom{r-1}{j}(1-x)^j x^{r-1-j}\frac{y^k}{k!} =$$

$$= 1 + \sum_{k=0}^{r-2}\sum_{j=k}^{r-2}\binom{j}{k}(1-x)^{k+1}x^{j-k}\frac{y^{k+1}}{(k+1)!},$$

because $\sum_{j=0}^{r-1}\binom{r-1}{j}(1-x)^j x^{r-1-j} = 1.$

Theorem 3 is obtained by changing the summation order on both sides of equality (7).

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REZIME

NEKI KOMBINATORNI IDENTITETI INSPIRISANI ZAKONOM ENERGIJE

Rezultati rada [1] i zakon o održanju energije nameću kombinatorne identitete (1),(4) i (5) čiji dokazi su bili neophodni za fizičku interpretaciju kvalitativnih i kvantitativnih poenti rada [1], a koji su dati u ovom radu.

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ONE ELEMENT KEY CIPHER

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ABSTRACT

From the 26-letter alphabet of the English language the letters x and z are dropped and replaced by the combination ks and cs. The remaining 24 letters are taken in some order as elements of the symmetric group S_4. An algorithm is described for changing the clear words (of a plain text) into the hidden ones. To make such a cipher only one letter as a key needs to be memorized (the group S_4 being automatically constructed starting with S_1). The encryption and decryption of this cipher is described and the measure of security of the cipher is given. A worked out example is also given.

1. Cryptography is usually not considered as a branch of the theory of formal languages. But there are many reasons to do so: There is a finite alphabet A and the set of "clear" words $w_1, w_2, \ldots$ over A is to be transformed into the set of "secret" words $u_1, u_2, \ldots$ (over the same alphabet A) by using some grammar g ("method of ciphering"). Usually the clear word w is transformed into the secret

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word u by replacing the letters of w by the letters of u starting with some key word u. This means that u is some gramatical function of w and of u_o:

(1) $$u = f_g(w, u_o) \quad .$$

The key u_o is, in fact, that part of the grammar g which is "invertible" and which means that, given u, u_o and g one can reconstruct w

(2) $$w = \phi_g(u, u_o)$$

2. We shall explain this by develloping an original group theoretic algorithm of ciphering in which the key is a letter u_o. As is always the case, and as it must be the case, deciphering is a very intricate job when the grammar g and the key u_o are unknown to the decoder.

3. We shall take the latin alphabet of 24 letters omitting the letters x and z which in the clear text have been previously replaced by the combinations "ks" and "cs" respectively.

4. In the symmetric group S_4 each element of the group is replaced by some letter out of our 24-letter alphabet A. We present here the table of composition of this group. This table can be automatically constructed by starting with S_2 or even with S_1 and therefore need not be memorized.

5. Let the clear word w (we consider the sentence a word) be composed of the letters $a_1, a_2, a_3, \ldots$, let the key letter be u_o and let the secret word u, corresponding to the word w, be composed of the letters $u_1, u_2, u_3, \ldots$ Then our ciphering grammar is given by the table

(3) $$\begin{aligned} u_1 &= u_o a_1 \;, \\ u_2 &= u_1 a_2 \;, \\ u_3 &= u_2 a_3 \;, \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

	1234 A 1	1243 B 2	1324 C 3	1342 D 4	1423 E 5	1432 F 6	2134 G 7	2143 H 8	2314 I 9	2341 J10	2413 K11	2431 L12	3124 M13	3142 N14	3214 O15	3241 P16	3412 Q17	3421 R18	4123 S19	4132 T20	4213 U21	4231 V22	4312 W23	4321 Y24
A 1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
B 2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
C 3	3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
D 4	4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
E 5	5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
F 6	6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	4	15	13	24	22	23	20	21	19
G 7	7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
H 8	8	7	14	13	20	19	2	1	16	15	22	21	4	3	19	9	24	23	6	5	12	11	18	17
I 9	9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
J10	10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
K11	11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
L12	12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
M13	13	19	7	20	8	14	15	21	1	22	2	16	9	23	2	24	4	10	11	17	5	18	6	12
N14	14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
O15	15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
P16	16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
Q17	17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
R18	18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
S19	19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
T20	20	14	19	8	13	7	22	26	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
U21	21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
V22	22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	17	15	1	9	3
W23	23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
Y24	24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

where ab is the composition of the letters a,b in the group S_4.

The deciphering is naturally given by the table

$$a_1 = u_o^{-1}u_1 ,$$
$$a_2 = u_1^{-1}u_2 ,$$
$$a_3 = u_2^{-1}u_3 ,$$
$$\cdots\cdots\cdots\cdots$$

Recalling that S_4 is a group, we have

THEOREM 1. *The grammar* g, *defined by* (3), *is the one-to-one mapping* $w \to u$ *(and is therefore invertible).*

6. As an example let us take the clear sentence

w = "suspe ndthe attac kimme diate lyyyy"

Let the one letter key be given by the letter E so that

$$u_o = E .$$

The first letter u_1 of the secret word U is $u_1 = W = ES$ where S is the first clear letter S.

The new key letter is now $u_1 = W$ and proceeding as formulated in (3) we get the secret word u:

u = "wnlht jrosn nrooi svswk dccwo rgrgc"

7. The measure of complexity of this cipher is given by

THEOREM 2. *As the measure* μ *of the complexity of the cipher described by* (3) *can be taken the number*

$$\mu = 24!\cdot 23\cdot 2$$

P r o o f. There are 24! combinations of associating 24 letters to the elements of S_4. There are 23 possibilities to select the key (the neutral element of S_4 is omitted as

the key for evident reasons) and there are 2 possibilities to use the key on the left or on the right side in the composition of elements of S_4. This measure is valid in the case that the grammar g, using S_4, is somehow known to the decoder. In the opposite case we do not know how to express the measure of complexity of our cipher (the upper limit for the complexity being here is $24^{\text{length } w}$ when g is supposed to be behind the cipher).

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REZIME

ŠIFRA SA JEDNOELEMENTNIM KLJUČEM

Iz azbuke od 26 slova engleskog jezika dva slova x i z zamenjena su respektivno sa ks i cs. Preostala 24 slova uzeta su nekim redom za elemente simetrične grupe S_4. Opisan je algoritam za pretvaranje jasnog teksta u skriveni u koju svrhu treba kao ključ pamtiti samo jedno slovo (dok se tabela za S_4 automatski dobija iz S_1). Šifrovanje i dešifrovanje za ovu šifru je opisano i jedan primer naveden. Mera sigurnosti šifre takodje je procenjena.

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THE INVERSE OF A TR-LATTICE NEED NOT BE A TR-LATTICE

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ABSTRACT.

A Tr-lattice is a finite lattice L with the following property: if L is isomorphic to the lattice of cyclic flats of a matroid M, then M is transversal. In this paper we give an example of a Tr-lattice L, such that the inverse lattice L^{-1} is not a Tr-lattice.

PRELIMINARIES

An n-set is a set of cardinality n.

The cardinality of a set X is denoted by "$|X|$".

The denotation "k·K" means "k copies of the set K".

We assume familiarity with the notions "lattice", "inverse lattice", "lattice isomorphism" and "chain" (type of lattice).

A matroid M on a finite set (the ground-set of M) S is an ordered pair (S,B), where B is a family of subsets

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of S, which satisfies the following "exchange" property:

$$(B_1, B_2 \in \mathcal{B} \ \wedge \ x \in B_1 \setminus B_2) \Rightarrow$$

$$\Rightarrow (\exists y)(y \in B_2 \setminus B_1 \ \wedge \ (B_1 \setminus x) \cup y \in \mathcal{B})$$

The sets of $\mathcal{B}$ are the <u>bases</u> of M.

Two matroids are <u>isomorphic</u> if there is a bijection between their ground-sets, which preserves the bases.

All (including trivial) subsets of bases are the <u>independent</u> sets of M, while the other subsets of S are <u>dependent</u>.

A <u>circuit</u> of M is a minimal dependent set.

A <u>loop</u> of M is a circuit of cardinality 1.

If X is a subset of S, then

$$\mathrm{rank}_M(X) = \max_{B \in \mathcal{B}} |X \cap B|$$

We simply write "rank(X)" when the matroid M is known. It is obvious that a subset X of S is independent if and only if rank(X) = |X|. We define

$$\mathrm{rank}(M) \stackrel{\mathrm{def}}{=} \mathrm{rank}(S)$$

and this number is known to be the common cardinality of all bases of M.

REMARK: The proofs of this and further unproved statements can be found in [6], unless another reference is cited.

The rank-function of the matroid M satisfies the following ("submodular") inequality for all subsets X and Y of S:

$$\mathrm{rank}(X \cup Y) + \mathrm{rank}(X \cap Y) \leq \mathrm{rank}(X) + \mathrm{rank}(Y)$$

A subset X of S is a <u>flat</u> of M if it satisfies

$$\mathrm{rank}(X \cup y) = \mathrm{rank}(X) + 1, \qquad \text{for all } y \in S \setminus X.$$

The closure of a subset X of S is the minimal flat of M, which contains X.

A flat X of M is cyclic if additionally it satisfies

$$\mathrm{rank}(X\setminus y) = \mathrm{rank}(X), \quad \text{for all } y \in X.$$

It is well-known ([4]) that all cyclic flats of a matroid M constitute a lattice (ordered by set-inclusion), which we call the CF-lattice of M. A matroid is uniquely (up to an isomorphism) determined by the family of its cyclic flats, accompanied by their ranks.

It is also known ([5]) that each finite lattice is the CF-lattice of a matroid.

The family $\{X \subseteq S \mid S\setminus X \in \mathcal{B}\}$ is known to be the family of bases of another, so-called, dual matroid M^* on S. It also holds that a set X is a cyclic flat of M if and only if the set $S\setminus X$ is a cyclic flat of M^* ([1]). A consequence of this fact is that the lattice, which is isomorphic to the CF-lattice of M^*, can be obtained by inversion of the lattice, which is isomorphic to the CF--lattice of M —— and conversely.

A coloop of M is a loop of M^*.

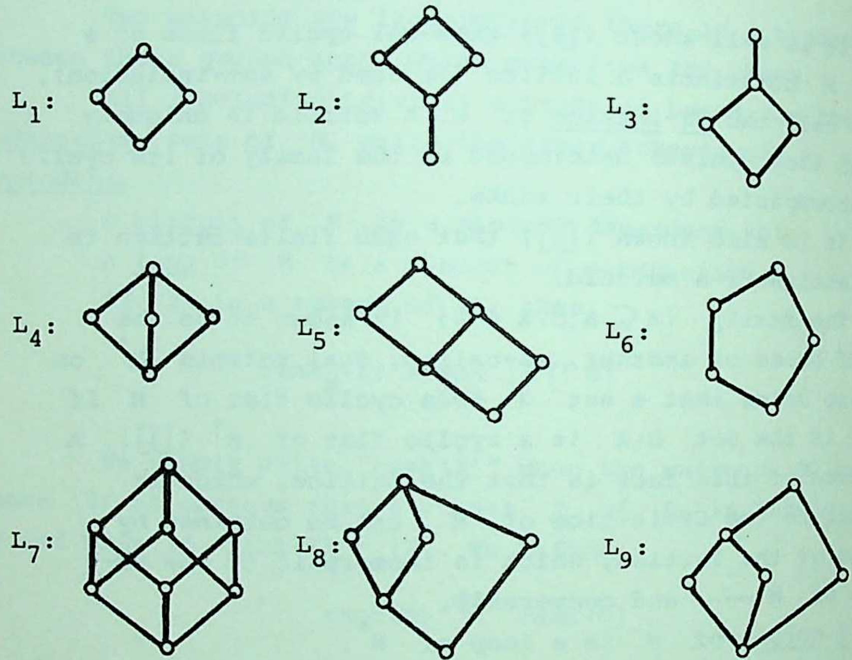
Let $\tau = \{T_1,\ldots,T_r\}$ be a finite family of finite sets. A set $X = \{x_1,\ldots,x_j\}$ is a partial transversal (= a system of distinct representatives of the sets) of τ if there is an injection π of the set $\{1,\ldots,j\}$ into the set $\{1,\ldots,r\}$, such that

$$x_i \in T_{\pi(i)}, \quad 1 \le i \le j$$

If $j = r$, then X is a transversal of τ. It is well-known that the maximal partial transversals of τ are the bases of a matroid M_τ. The family τ is a transversal representation of the matroid M_τ. A matroid is transversal if it has a transversal representation. Each transversal matroid has a transversal representation in which the bases are transversals.

A <u>Tr-lattice</u> is a finite lattice L, which satisfies: If L is isomorphic to the CF-lattice of a matroid M, then M is transversal.

We shall introduce nine special finite lattices $L_1, L_2, \ldots, L_9$, which are given in the following table:



INTRODUCTION

It is proved in paper [2] that all chains (of arbitrary length) and the lattice L_1 are Tr-lattices. We also know that the lattices L_2 and L_3 are Tr-lattices. However, the very simple lattice L_4 is not a Tr-lattice.

The matroids with the CF-lattices isomorphic to L_5, L_6 and L_7 are rather numerous in the catalogue [3] and all such examples are transversal matroids. We conjecture for that reason that the lattices L_5, L_6 and L_7 are also Tr-lattices.

The following example from the same catalogue attrac-

ted our particular attention:

All 37 non-isomorphic matroids on at most 8 elements, the CF-lattices of which are isomorphic to L_8, ——— are transversal. However, among the 37 dual matroids, the CF-lattices of which are isomorphic to $L_9 = L_8^{-1}$ (the CF-lattices of mutually dual matroids, when considered in a pure lattice - theoretical sense, are mutually inverted), ——— there are only 14 transversal matroids. This is the "smallest" example of this kind that we know and it motivated the assertion, which coincides with the title of this paper. Such an assertion is closely connected with (but it by no means directly follows from) the well-known fact that the dual of a transversal matroid need not be a transversal matroid.

In this paper we shall prove that the lattice L_8 is a Tr-lattice (this is the main difficulty) and we shall give an example of a non-transversal matroid with the CF-lattice isomorphic to $L_9 = L_8^{-1}$.

RESULTS

We prove two lemmas primarily:

LEMMA 1. *The closure of a circuit of a matroid is a cyclic flat.*

P r o o f. If we suppose that the closure of a circuit C is a non-cyclic flat X, then there exists $e \in X$ such that $\mathrm{rank}(X\setminus e) = \mathrm{rank}(X) - 1$.

If $e \in C$, then

$$\mathrm{rank}(X) = \mathrm{rank}(C) = \mathrm{rank}(C\setminus e) \le \mathrm{rank}(X\setminus e) = \mathrm{rank}(X) - 1$$

a contradiction.

If $e \in X\setminus C$, then we have the same contradiction; the only difference is in that the inequality $\mathrm{rank}(C) \le \mathrm{rank}(X\setminus e)$ follows directly. □

LEMMA 2. *Let* M *be a matroid on* S. *A subset* X *of* S *is independent in* M *if and only if* $|X \cap F| \leq \mathrm{rank}(F)$ *for each cyclic flat* F *of* M.

P r o o f. Let F be a cyclic flat of M such that $|X \cap F| > \mathrm{rank}(F)$. Then we also have $|X \cap F| > \mathrm{rank}(X \cap F)$, that is, the set $X \cap F$ is dependent in M. It follows that the superset X is also dependent in M.

Conversely, let X be a subset of S, which is dependent in M. Then X contains a circuit C. The closure of the circuit C is by Lemma 1 a cyclic flat F. Then by

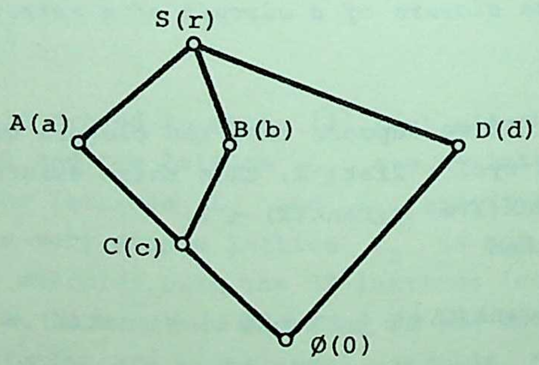
$$|X \cap F| \geq |C| > |C| - 1 = \mathrm{rank}(C) = \mathrm{rank}(F)$$

we have that the cyclic flat F of M satisfies $|X \cap F| > \mathrm{rank}(F)$. □

We shall proceed with proving the main theorem of this paper:

THEOREM 1. L_8 *is a Tr-lattice.*

P r o o f. Let M (on S) be an arbitrary matroid having the CF-lattice isomorphic to L_8. We denote the cyclic flats of M and their ranks (in brackets) as follows:



REMARK: We may assume that M has no loops or coloops, since they do not influence transversality (the coloops appear in each base of a matroid and the loops in none). It is for this reason that we may assume that the minimal element in L_8 is the empty set (necessarily of rank 0), while the maximal element coincides with the ground-set S of M.

We shall prove that the matroid M is transversal, by exhibiting its explicit transversal representation. Namely, we claim that the family

$$\Phi = \{(r - a)\cdot(S\setminus A),\ (r - b)\cdot(S\setminus B),$$
$$(a + b - r - c)\cdot(S\setminus C),\ (r - d)\cdot(S\setminus D),\ (c + d - r)\cdot S\}$$

is a transversal representation of M.(*)

To prove this, it suffices to prove the following two statements:

(i) each dependent r-subset X of S is not a transversal of Φ

(ii) each base of M is a transversal of Φ.

P r o o f of (i): Following Lemma 2, we conclude that $|X \cap F| > \text{rank}(F)$ for some $F \in \{A,B,C,D\}$. If $F = A$, then X contains less than $r-a$ elements in the set $S\setminus A$. Thus X does not contain a transversal of the subfamily $\{(r-a)\cdot(S\setminus A)\}$ of Φ, which implies that X is not a transversal of Φ. We apply a very similar reasoning if $F = B$ or $F = D$. If $F = C$, then we observe that

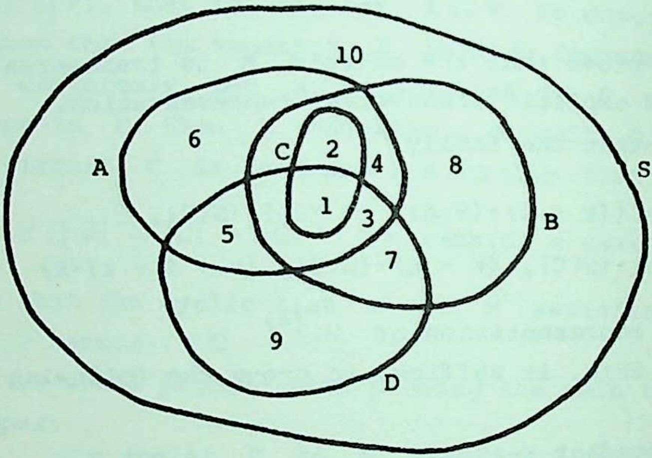
$$(r - a) + (r - b) + (a + b - r - c) = r - c$$

and that $S\setminus A \subset S\setminus C$ and $S\setminus B \subset S\setminus C$.
Since X contains less than $r-c$ elements in $S\setminus C$, it does

(*) *this general transversal representation was "experimentally" derived by using* [3].

not contain a transversal of the subfamily $\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-c)\cdot(S\setminus C)\}$ of Φ.

P r o o f of (ii): The set S is partitioned by its subsets A,B,C,D into ten pairwise disjoint subsets, which is shown by the following diagram:



The number i in the diagram corresponds to the subset denoted by O_i, $1 \le i \le 10$.

Let X be a base of M and let $x_i = |X \cap O_i|$, $1 \le i \le 10$. We derive and numerate nine inequalities and one equality, which are satisfied by the numbers $x_1,\dots,x_{10}$:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le a \tag{1}$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 \le b \tag{2}$$
$$x_1 + x_2 \le c \tag{3}$$
$$x_1 + x_3 + x_5 + x_7 + x_9 \le d \tag{4}$$
$$x_1 + x_2 + x_3 + x_4 \le a + b - r \tag{5}$$
$$x_1 + x_3 + x_5 \le a + d - r \tag{6}$$
$$x_1 + x_3 + x_7 \le b + d - r \tag{7}$$
$$x_1 \le c + d - r \tag{8}$$
$$x_1 + x_3 \le a + b + d - 2r \tag{9}$$
$$x_1 + x_2 + \dots + x_{10} = r \tag{10}$$

The first four inequalities, follow from the fact that the set X is independent, by the use of Lemma 2 applied to the cyclic flats A,B,C,D respectively. The next four inequalities are derived by means of

$$|X \cap (F \cap G)| \leq \text{rank}(F \cap G) \leq \text{rank}(F) + \text{rank}(G) - \text{rank}(F \cup G)$$

where $\{F,G\}$ is equal to $\{A,B\}$, $\{A,D\}$, $\{B,D\}$, $\{C,D\}$ respectively. In a similar way (9) is obtained, but the last inequality (the submodular law) should be applied twice, e.g. separately to the pairs $\{A,B\}$ and $\{A \cap B,D\}$. Finally, (10) is equivalent to $|X| = r$.

REMARK: The right-hand sides of the inequalities (5) - (9) are non-negative, otherwise the submodular law for the rank-function of the matroid M would be violated. We point out that this nonnegativity of different coefficients, which we adjoin to the sets of Φ in the course of proving, follows either from the relations (1) - (10) or from some additional assumptions.

We are going to show that X is a transversal of Φ. This will be done step-by-step. We shall gradually make the sets $X \cap O_i$, $1 \leq i \leq 10$, to be some pairwise disjoint partial transversals of Φ. The elements of a set $X \cap O_i$ can represent only those sets in Φ, which are supersets of of O_i. We list such supersets for each O_i, $1 \leq i \leq 10$:

O_1: S ; O_2: $S \setminus D, S$; O_3: $S \setminus C, S$;
O_4: $S \setminus C, S \setminus D, S$; O_5: $S \setminus B, S \setminus C, S$; O_6: $S \setminus B, S \setminus C, S \setminus D, S$;
O_7: $S \setminus A, S \setminus C, S$; O_8: $S \setminus A, S \setminus C, S \setminus D, S$;
O_9: $S \setminus A, S \setminus B, S \setminus C, S$; O_{10}: $S \setminus A, S \setminus B, S \setminus C, S \setminus D, S$

We use the abbreviation "x_i <u>is covered</u>" to denote that the elements of the set $X \cap O_i$ are appropriately represented by some sets of Φ ("appropriately" means: by some sets which have not been already used). Our proof is over at the moment when all x_i - s, $1 \leq i \leq 10$, are covered.

x_1 can be directly covered by (8).

x_2 can be covered by reason of the inequality

$$x_2 \le (r - d) + (c + d - r - x_1)$$

which is equivalent to (3) (in what follows we just put down the number of the equivalent inequality). Namely, for covering x_2 we may use only some of $r-d$ sets $S\setminus D$ and the remaining $c+d-r-x_1$ sets S in Φ. However, we give an "advantage" to the sets $S\setminus D$. We use some of the sets S only if all the sets $S\setminus D$ are exhausted. The reason is that the sets S are more "universal": they can be used for covering all x_i, $3 \le i \le 10$. In what follows, we shall always give an adventage in covering to less universal sets of Φ.

We observe that the sets $S\setminus C$ can also cover all x_i, $3 \le i \le 10$. Therefore we need not make any difference between the sets S and $S\setminus C$ in the rest of our proof. Let "W" denote "any of the sets S and $S\setminus C$". Then we replace $(a+b-r-c)\cdot(S\setminus C)$ and $(c+d-r-x_1)\cdot S$ in Φ by $(a+b+d-2r-x_1)\cdot W$, while the new families of supersets for O_i, $3 \le i \le 10$, are:

O_3: W; O_4: $S\setminus D$,W; O_5: $S\setminus B$,W; O_6: $S\setminus B$,$S\setminus D$,W;

O_7: $S\setminus A$,W; O_8: $S\setminus A$,$S\setminus D$,W; O_9: $S\setminus A$,$S\setminus B$,W;

O_{10}: $S\setminus A$,$S\setminus B$,$S\setminus D$,W

It should be stressed that the rest of our proof will include a kind of branching process. The first branching depends on whether the sets $S\setminus D$ are exhausted while covering x_2 or not. We denote the first possibility by <1> and the second one by <2>. We shall also use the denotations "1" and "2" in further branchings, depending on whether the less universal sets are exhausted or not. Thus each particular branch of the branching will be denoted by a binary vector (with components 1,2) in brackets "< >". This vector will be followed by the condition

(an inequality in the brackets "( )"), which corresponds to the last branching. After that the remaining subfamily of ϕ will be listed and one or more relations (equivalent to some of the relations (1)-(10)), which correspond to the coverings on that branch.

$$\langle 1\rangle : (x_2 \geq r - d)$$

After covering x_2, the following subfamily remains from Φ for further coverings:

$$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2)\cdot W\}$$

We observe that after eliminating the sets $S\setminus D$, some families of supersets coincide as follows:

$$O_3 \cup O_4 : W\ ; \quad O_5 \cup O_6 : S\setminus B, W\ ;$$

$$O_7 \cup O_8 : S\setminus A, W\ ; \quad O_9 \cup O_{10} : S\setminus A, S\setminus B, W$$

We cover together the corresponding x_i - S:

$$x_3+x_4 \leq a+b-r-x_1-x_2 \qquad , \quad (5)$$

The next remaining subfamily is:

$$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$$

$$x_5+x_6 \leq (r-b) + (a+b-r-x_1-x_2-x_3-x_4) \qquad , \quad (1)$$

The next branching depends on the way of covering x_5 and x_6:

$$\langle 1,1\rangle : (x_5+x_6 \geq r-b)$$

The remaining subfamily is:

$$\{(r-a)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

We should further cover x_i, $7 \leq i \leq 10$, by using, solely, the sets $S\setminus A$ and W. This is possible on the basis of

$$x_7+x_8+x_9+x_{10} = (r-a) + (a-x_1-x_2-x_3-x_4-x_5-x_6) \quad , \qquad (10)$$

$\langle 1,2\rangle$: $(x_5+x_6 < r-b)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b-x_5-x_6)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$

We further have

$$x_7+x_8 \leq (r-a) + (a+b-r-x_1-x_2-x_3-x_4) \quad , \qquad (2)$$

and regardless of the way of covering x_7 and x_8:

$$x_9+x_{10} = ((r-a) + (a+b-r-x_1-x_2-x_3-x_4) - (x_7+x_8)) + \\ + (r-b-x_5-x_6) \quad , \qquad (10)$$

covers x_9 and x_{10}.

$\langle 2\rangle$: $(x_2 < r-d)$

The covering x_2 does not exhaust the sets $S\setminus D$ and after covering x_3 with the sets W (it is possible because of

$$x_3 \leq a+b+d-2r-x_1 \quad , \qquad (9)\ ),$$

the remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (r-d-x_2)\cdot(S\setminus D)\ ,$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We cover primarily x_4:

$$x_4 \leq (r-d-x_2) + (a+b+d-2r-x_1-x_3) \quad , \qquad (5)$$

We have the following branching, depending on the way of covering x_4:

$\langle 2,1\rangle$: $(x_4 \geq r-d-x_2)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$

This is the same family as in case <1> and we can cover x_i, $5 \le i \le 10$, in the same way as before (the only difference between cases <1> and <2,1> is in the way of covering x_4: in the first case only the sets W are used, while in the second one the sets $S\setminus D$ are primarily exhausted).

<2,2> : $(x_4 < r-d-x_2)$

The remaining subfamily after covering x_i, $1 \le i \le 4$, is

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (r+d-x_2-x_4)\cdot(S\setminus D),$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We can cover x_5 on the basis of

$$x_5 \le (r-b) + (a+b+d-2r-x_1-x_3) \quad , \tag{6}$$

The way of covering this determines the next branching:

<2,2,1> : $(x_5 \ge r-b)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-d-x_2-x_4)\cdot(S\setminus D),\ (a+d-r-x_1-x_3-x_5)\cdot W\}$

Since the sets $S\setminus B$ are exhausted, the scheme of supersets for further coverings is:

O_6 : $S\setminus D, W$; $O_7 \cup O_9$: $S\setminus A, W$

$O_8 \cup O_{10}$: $S\setminus A, S\setminus D, W$

We proceed with covering x_6:

$$x_6 \le (r-d-x_2-x_4) + (a+d-r-x_1-x_3-x_5) \quad , \tag{1}$$

The next branching depends on whether the sets $S\setminus D$ are exhausted or not:

$$<2,2,1,1> : (x_6 \geq r-d-x_2-x_4)$$

The following subfamily remains from Φ for further coverings:

$$\{(r-a)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

Since the sets $S\setminus A$ and W are the supersets of all sets O_i, $7 \leq i \leq 10$, it follows that the equality:

$$x_7+x_8+x_9+x_{10} = (r-a) + (a-x_1-x_2-x_3-x_4-x_5-x_6) \quad , \qquad (10)$$

completes the necessary coverings in this case.

$$<2,2,1,2> : (x_6 < r-d-x_2-x_4)$$

After covering x_i, $1 \leq i \leq 6$, the following subfamily remains from ϕ:

$$\{(r-a)\cdot(S\setminus A),\ (r-d-x_2-x_4-x_6)\cdot(S\setminus D),\ (a+d-r-x_1-x_3-x_5)\cdot W\}$$

Simultaneously we cover x_7 and x_9 with

$$x_7+x_9 \leq (r-a) + (a+d-r-x_1-x_3-x_5) \quad , \qquad (4)$$

Without regard to the way of this covering, the coverings are completed by

$$x_8+x_{10} = ((r-a) + (a+d-r-x_1-x_3-x_5) - (x_7+x_9)) + \\ + (r+d-x_2-x_4-x_6) \quad , \qquad (10)$$

for any of the sets $S\setminus A, S\setminus D$ and W can be used for covering x_8 and x_{10}.

$$<2,2,2> : (x_5 < r-b)$$

The remaining subfamily after covering x_i, $1 \leq i \leq 5$, is

$\{(r-a)\cdot(S\setminus A),\ (r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We cover primarily x_7 (before x_6), for it can be covered only by two kinds of sets, $S\setminus A$ and W:

$$x_7 \leq (r-a) + (a+b+d-2r-x_1-x_3) \quad , \qquad (7)$$
$$\langle 2,2,2,1\rangle : (x_7 \geq r-a)$$

The following subfamily remains:

$$\{(r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),\ (b+d-r-x_1-x_3-x_7)\cdot W\}$$

The corresponding scheme of supersets is:

$$O_8 : S\setminus D, W \ ; \quad O_9 : S\setminus B, W \ ; \quad O_6 \cup O_{10} : S\setminus B, S\setminus D, W$$

The next covering and branching are related to x_8:

$$x_8 \leq (r-d-x_2-x_4) + (b+d-r-x_1-x_3-x_7) \quad , \qquad (2)$$
$$\langle 2,2,2,1,1\rangle : (x_8 \geq r-d-x_2-x_4)$$

It remains

$$\{(r-b-x_5)\cdot(S\setminus B),\ (b-x_1-x_2-x_3-x_4-x_7-x_8)\cdot W\}$$

Since x_6, x_9 and x_{10} can be covered by any of the sets $S\setminus B$ and W, the covering is completed by

$$x_6+x_9+x_{10} = (r-b-x_5) + (b-x_1-x_2-x_3-x_4-x_7-x_8) \ , \qquad (10)$$
$$\langle 2,2,2,1,2\rangle : (x_8 < r-d-x_2-x_4)$$

The remaining subfamily is

$$\{(r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4-x_8)\cdot(S\setminus D),\ (b+d-r-x_1-x_3-x_7)\cdot W\}$$

We cover x_9 with

$$x_9 \leq (r-b-x_5) + (b+d-r-x_1-x_3-x_7) \quad , \qquad (4)$$

Disregarding the relation between x_9 and $r-b-x_5$, x_6 and x_{10} can be covered on the basis of

$$x_6+x_{10} = ((r-b-x_5) + (b+d-r-x_1-x_3-x_7) - x_9) + \\ + (r-d-x_2-x_4-x_8) \quad , \qquad (10)$$

for all three kinds of sets: $S\setminus B$, $S\setminus D$ and W, can be used for their covering.

$<2,2,2,2> : (x_7 < r-a)$

The following subfamily remains from ϕ:

$$\{(r-a-x_7)\cdot(S\setminus A),\ (r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),\\ (a+b+d-2r-x_1-x_3)\cdot W\}$$

We recall that the corresponding scheme of supersets is:

$$O_6 : S\setminus B, S\setminus D, W \quad ; \qquad O_8 : S\setminus A, S\setminus D, W$$

$$O_9 : S\setminus A, S\setminus B, W \quad ; \qquad O_{10} : S\setminus A, S\setminus B, S\setminus D, W$$

that is, each of x_6, x_8, x_9, x_{10} can be covered by using three (respectively four) different kinds of sets. We cover primarily x_6 by use of

$$x_6 \le (r-b-x_5) + (r-d-x_2-x_4) + (a+b+d-2r-x_1-x_3) \ , \qquad (1)$$

This time our branching is somewhat different: it depends on whether the BOTH less universal kinds of sets, $S\setminus B$ and $S\setminus D$, are exhausted or not.

$<2,2,2,2,1> : (x_6 > (r-b-x_5) + (r-d-x_2-x_4))$

The remaining subfamily is

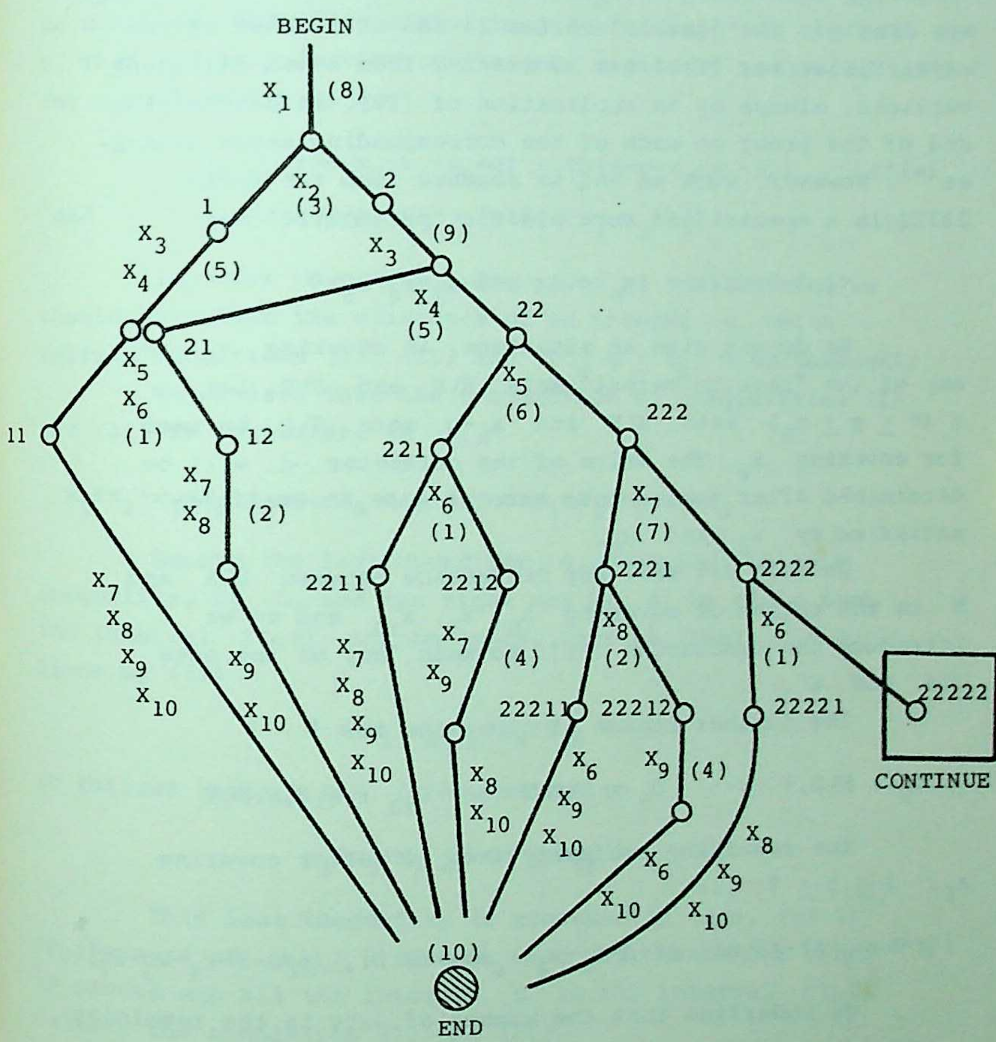
$$\{(r-a-x_7)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

Since x_8, x_9 and x_{10} can be covered by both $S\setminus A$ and W, it follows that the equality

$$x_8+x_9+x_{10} = (r-a-x_7) + (a-x_1-x_2-x_3-x_4-x_5-x_6), \qquad (10)$$

completes the coverings.

We give a diagram of the "branching part" of our proof:



The coverings are denoted by the coresponding x_i-s. Each non-denoted vertex corresponds to a realized single covering step. The vertices denoted by binary vectors (without brackets and commas) follow the corresponding branchings. In both cases the inequalities used for the coverings are denoted. The "double" vertex 21 can be reached in two ways. The vertex "END" can be reached from seven different vertices, always by an application of (10). It denotes the end of the proof on each of the corresponding seven branches(*). However, such an end is reached from the vertex 22222 in a special and more elastic "parametric" way:

$$\langle 2,2,2,2,2\rangle : (x_6 \leq 2r-b-d-x_2-x_4-x_5)$$

We do not give an adventage, in covering x_6, to any of the "less universal" sets $S\backslash B$ and $S\backslash D$. Let g $(0 \leq g \leq x_6)$ sets $S\backslash B$ and x_6-g sets $S\backslash D$ be used for covering x_6. The value of the parameter g will be determined after taking into account some inequalities satisfied by x_8 and x_9.

We need not make any difference between $S\backslash A$ and W in the course of covering x_8, x_9, x_{10} and so we introduce the denotation "Y" to mean "any of the sets $S\backslash A$ and W".

The further scheme of supersets is:

$$O_8 : S\backslash D, Y \quad ; \qquad O_9 : S\backslash B, Y \quad ; \qquad O_{10} : S\backslash D, S\backslash B, Y$$

The remaining subfamily from Φ after covering x_i, $1 \leq i \leq 7$, is:

$$\{(r-b-x_5-g)\cdot(S\backslash B),\ (r-d-x_2-x_4-x_6+g)\cdot(S\backslash D),\ (b+d-r-x_1-x_3-x_7)\cdot Y\}$$

We underline that the number of sets in the remaining

(*) *We observe that (10) completes the proof whenever all the kinds of the remaining sets in* Φ *can be used for the remaining coverings.*

subfamily is equal to the number of still uncovered elements in X, that is, $x_8+x_9+x_{10}$. The sets $S\backslash B$ cannot be used for covering x_8 and the same holds for the sets $S\backslash D$ and x_9. Since the sets $S\backslash D$, respectively the sets $S\backslash B$, have an advantage in covering x_8, respectively x_9, we conclude that there cannot be a "deficiency" of the sets Y. Thus the only further conditions, necessary and sufficient for a full covering, are

$$x_8 \leq (r-d-x_2-x_4-x_6+g) + (b+d-r-x_1-x_3-x_7) \qquad \text{(I)}$$

and

$$x_9 \leq (r-b-x_5-g) + (b+d-r-x_1-x_3-x_7) \qquad \text{(II)}$$

In order to complete the proof of our Theorem, we should establish the existence of an integer g, which fulfils conditions (I), (II) and $0 \leq g \leq x_6$ simultaneously.

We observe that the conjunction of inequalities (I) and (II) is equivalent to

$$x_1+x_2+x_3+x_4+x_6+x_7+x_8-b \leq g \leq d-x_1-x_3-x_5-x_7-x_9$$

Denote the left bound for g, contained in this inequality, by L, and the right one by R. We claim that the interval $[L,R]$ is non-empty, that is, that $L \leq R$. Since by (10)

$$R = d-r+x_2+x_4+x_6+x_8+x_{10}$$

it follows that $L \leq R$ is equivalent to

$$x_1+x_3+x_7 \leq b+d-r+x_{10}$$

This last inequality is necessarily true, for it is (because of $x_{10} \geq 0$) weaker than (7). Hence our claim is proved and all the integers g in the interval $[L,R]$ satisfy the conditions (I) and (II).

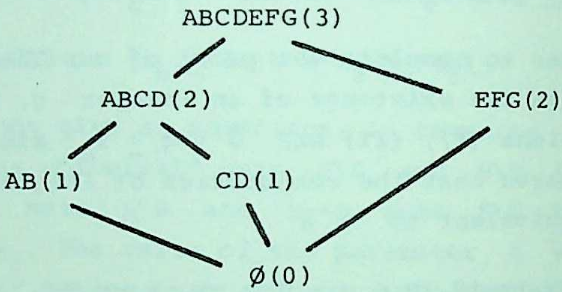
We should only prove that at least one of these g belongs to the interval $[0,x_6]$, that is, that

$$[0,x_6] \cap [L,R] \neq \emptyset$$

To prove this, it suffices to show that $O \leq R$ and $L \leq x_6$. We see immediately that the first of the last two inequalities is equivalent to (4), while the second is equivalent to (2). Thus the proof of Theorem 1 is completed.□

THEOREM 2. *There exists a non-transversal matroid with the CF-lattice isomorphic to* L_9.

P r o o f. Let M be a rank 3 matroid on the set $\{A,B,C,D,E,F,G\}$ with the following CF-lattice:



The cyclic flats of M are denoted without brackets and commas; we shall adopt this convention for all sets in the rest of our proof. The numbers in brackets denote the ranks of the corresponding cyclic flats; the submodular law for the rank-function is not violated.

It is obvious that the CF-lattice of M is isomorphic to L_9. We are going to prove that M is not a transversal matroid.

Suppose, on the contrary, that M has a transversal representation $\tau = \{T_1,T_2,T_3\}$. Since $\mathrm{rank}(A) = \mathrm{rank}(B) = \mathrm{rank}(C) = \mathrm{rank}(D) = \mathrm{rank}(AB) = \mathrm{rank}(CD) = 1$, while $\mathrm{rank}(ABCD) = 2$, it follows that the elements A and B appear (together) in only one of the sets T_1,T_2,T_3, and so do the elements C and D, but it is not true that all the four elements A,B,C,D appear in the same set of τ. We may assume, without any loss of generality, that

$$A,B \in T_1 \setminus (T_2 \cup T_3) \qquad \text{and}$$
$$C,D \in T_2 \setminus (T_1 \cup T_3)$$

Let (W_1, W_2, W_3) be an arbitrary permutation of (T_1, T_2, T_3). Since $\mathrm{rank}(EF) = 2$, it follows that the set EF is a partial transversal of τ. We may assume that $E \in W_1$ and $F \in W_2$. We claim that $EFG \cap W_3 = \emptyset$.

The fact that $\mathrm{rank}(EFG) = 2$ implies that $G \notin W_3$ and this gives $G \in W_1 \cup W_2$ (because of $\mathrm{rank}(G) = 1$). If $G \in W_1$, then $\mathrm{rank}(EFG) = 2$ implies $E \notin W_3$. It follows that the rank 2 set EG is a transversal of $\{W_1, W_2\}$, which implies that $F \notin W_3$. If $G \in W_2$, then the proof of the claim is analogously completed.

If $W_3 \equiv T_3$, then the set T_3 is empty, which is a contradiction with $\mathrm{rank}(M) = 3$. Now suppose that $W_3 \equiv T_1$. Then $T_1 = AB$ and the base CEF is not a transversal of τ, a contradiction. The assumption $W_3 \equiv T_2$ and the base AEF lead to a similar contradiction. We conclude that τ cannot exist.□

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REZIME

INVERZNA MREŽA TR-MREŽE
NE MORA BITI TR-MREŽA

Tr-mreža je konačna mreža L, koja ima sledeće svojstvo: Ako je L izomorfna mreži cikličkih potprostora nekog matroida M, onda je matroid M transverzalan. U ovom radu dajemo primer Tr-mreže L, takve da inverzna mreža L^{-1} nije Tr-mreža.

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NOTE ON THE STANDARD MATRIX REPRESENTATION OF A MATROID

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ABSTRACT

The main result of this paper is: If $[I_k, A]$ is the SMR of a matroid with respect to a base B, then rank (A) $\geq$ C(B), where C(B) denotes the maximal number of pairwise disjoint fundamental circuits with respect to B.

INTRODUCTION.

The matroid theory terminology and results used in this paper conform to standard literature (see for example [3,4]).

Let E be a finite set, and M:=M(E,r) a <u>matroid</u> on E with r as the <u>rank function</u> ($r : 2^E \to \mathbb{N}$, where 2^E is the power set of E, and $\mathbb{N}$ the set of non-negative integers). A subset $S \subseteq E$ is called <u>independent</u> if $r(S) = |S|$, where $|S|$ denotes the cardinality of S. Let F(M) be the family of independent sets of M. A <u>basis</u> of M is a maximal independent subset of E. A subset of E which is not independent is called <u>dependent</u>, and a minimal dependent subset of E is a

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circuit. We shall denote by $C(M)$ the family of circuits of M. A circuit of cardinality 1 is a loop of M. For every $S \subseteq E$, $\bar{S}$ denotes the span of S, i.e. $\bar{S} = \{e \in E : r(S \cup \{e\}) = r(S)\}$.

If M is a matroid with $k = r(E)$, let $B = \{e_1, e_2, \ldots, e_k\}$ be a basis of M and $E - B = \{f_1, f_2, \ldots, f_m\}$. If M is representable over a field $\mathbb{F}$ (see [1]), it will have a standard matrix representation (see also [2]), with respect to the basis B, of the of the form $R = [I_k, A]$, where I_k is the identity matrix of order k and A is a $k \times m$ matrix with entries belonging to $\mathbb{F}$. If $E_1, E_2, \ldots, E_k$, $F_1, F_2, \ldots, F_m$ denote the column vectors of R, and the map σ, defined by $\sigma(e_i) = E_i$, $i=1, 2, \ldots, k$ and $\sigma(f_i) = F_i$, $i=1,2,\ldots,m$, is a representation of M over $\mathbb{F}$, in the sense that a subset S of E belongs to $F(M)$ if and only if the corresponding vectors of $\sigma(S)$ are linearly independent over $\mathbb{F}$ (see [3]).

If B is any basis of M and $\{f_1, f_2, \ldots, f_m\} = E - B$, then there exists a unique $C_i \in C(M)$, containing f_i and, otherwise, elements of B only. This circuit C_i is called fundamental with respect to f_i and B and will be denoted by $C(f_i, B)$. For a fixed basis B of a representable matroid M, we denote by $c(B)$ the maximal number of pairwise disjoint fundamental circuits with respect to $E - B$ and B. The main result of this paper is the following:

THEOREM. *If* $R = [I_k, A]$ *is the standard matrix representation with respect to* B, *then rank* $\operatorname{rank}(A) \geq c(B)$.

The key lemma. Let B be a fixed basis of M and $e \in E$ arbitrarily chosen. Obviously, $e \in \bar{B}$. Then we can consider the family of sets $D(e,B) = \{S \subseteq B : e \in \bar{S}\}$.

We shall denote by $B(e)$ a minimal (with respect to the inclusion of sets) element of $D(e,B)$. It can be checked that

$$\begin{aligned} &\text{if} \quad r(\{e\}) = 0, && \text{then } B(e) = \emptyset, \\ &\text{if} \quad e \in B, && \text{then } B(e) = \{e\}. \end{aligned}$$

Suppose that $e \notin B$, and let $C(e,B)$ be the fundamental circuit with respect to e and B. Obviously, by definition, $B(e) \cup \{e\}$ is a circuit of M contained in $B \cup \{e\}$, and therefore $B(e) \cup \{e\} = C(e,B)$. Thus, $B(e)$ is uniquely defined. Throughout, we shall make use of the following fundamental properties of the matroids:

(P1) If $C_1, C_2 \in C(M)$ such that $C_1 \subseteq C_2$, then $C_1 = C_2$.

(P2) If C_1, C_2 are two distinct circuits of M such that $e \in C_1 \cap C_2$, then there exists $C_{12} \in C(M)$ with $C_{12} \subseteq (C_1 \cup C_2) - \{e\}$.

(P3) A subset of E is independent if and only if it does not contain a circuit.

(P4) A subset of E is independent if and only if it is contained in a basis.

LEMMA. *Let* B *be a fixed basis of* M *and* $F \subseteq E$. *If* F *does not contain any loop and* $B(e) \cap B(f) = \emptyset$ *for every distinct* $e, f \in F$, *then* $F \in F(M)$.

P r o o f. Let $F = \{e_1, e_2, \ldots, e_t\}$. If $F \subseteq B$, then the theorem is trivial, by (P4). Thus, two cases must be considered:

(a) $F \cap B = \emptyset$,

(b) $F = H \cup G$, $H \cap B = \emptyset$, $G \subseteq B$ and $G \neq \emptyset$.

We prove now the lemma in case (a). Let us denote $B(e_i) = B_i$. We have seen above that $B_i \cup \{e_i\} = C(e_i, B)$.

Suppose that $F \notin F(M)$. By (P3), there exists $C \in C(M)$ such that $C \subseteq F$. Without loss of generality, we can consider C of the form $C = \{e_1, e_2, \ldots, e_s\}$. By (P2), there exists a circuit $C_1 \subseteq (C \cup C(e_1, B)) - \{e_1\}$. If $C_1 \subseteq C(e_1, B)$, then $C_1 = C(e_1, B)$ by (P1), and this is impossible. Thus, $C_1 \subseteq (B_1 \cup C) - \{e_1\}$ and $C_1 \neq C$. Similarly, there exists $C_2 \in C(M)$ with $C_2 \subseteq (B_2 \cup C) - \{e_2\}$ and $C_2 \neq C$. Suppose that $e_1 \in C_2$. By (P2), there exists $\tilde{C} \in C(M)$

such that $\tilde{C} \subseteq (C \cup C_2) - \{e_1\}$, $\tilde{C} \neq C_2$. Since $B_1 \cap B_2 = \emptyset$ (by hypothesis), we then have $\tilde{C} \neq C_1$. If $e_2 \in \tilde{C} \cap C_1$, then, by (P2), there exists $\tilde{\tilde{C}} \in C(M)$ with $\tilde{\tilde{C}} \subseteq (\tilde{C} \cup C_1) - \{e_1, e_2\}$. Therefore, there exists $C_o \in C(M)$ such that $C_o \subseteq (B_1 \cup B_2 \cup C) - \{e_1, e_2\}$.

On the other hand, $C_3 = B_3 \cup \{e_3\}$ is a circuit of M and $C_3 \neq C_o$. If $e_3 \in C_o$, then, by (P2), there exists $C_o' \in C(M)$ with $C_o' \subseteq (C_o \cup C_3) - \{e_1, e_2, e_3\}$. Thus, there exists C_o (if $e_3 \in C_o$, we take $C_o := C_o'$) in $C(M)$ such that $C_o \subseteq (B_1 \cup B_2 \cup B_3 \cup C) - \{e_1, e_2, e_3\}$. Repeating the above a finite number of times (s-3 times), thus, gives rise to a circuit C_o with $C_o \subseteq B_1 \cup B_2 \cup \ldots \cup B_s$, i.e. a subset of B contains a circuit, contradicting (P3) and (P4). Hence, $F \in F(M)$.

Now, we shall prove the lemma in case (b). Suppose that $F \notin F(M)$, i.e. by (P3), there exists $C \in C(M)$ such that $C \subseteq F$. Obviously, C must contain at least an element of H as otherwise $G \notin F(M)$, which is in contradiction with (P4). Considering C of the form $C = \{e_1, e_2, \ldots, e_s\} \cup T$ with $\{e_1, e_2, \ldots, e_s\} \subseteq H$ and $T \subseteq G$, and repeating the above judgement (as in case (a)) for the set $\{e_1, e_2, \ldots, e_s\}$, we obtain a similar contradiction: a subset of B contains a circuit. Therefore, the lemma is entirely proved. (QED).

COROLLARY. *Let* B *be a fixed basis of* M, *and* $\{f_{i_1}, f_{i_2}, \ldots, f_{i_t}\} \subseteq E - B$ *such that* $C(f_{i_j}, B)$ *are pairwise disjoint. Then* $\{f_{i_1}, f_{i_2}, \ldots, f_{i_t}\} \in F(M)$.

P r o o f. It readily follows form tha lemma (QED).

Proof of the theorem. Let M be a representable matroid over the field $\mathbb{F}$, B a fixed basis of M and $R = [I_k, A]$ the standard matrix representation with respect to B. By the above corollary, it follows that the column vectors

$F_{i_1}, F_{i_2}, \ldots, F_{i_t}$, are linearly independent over $\mathbb{F}$, i.e. $\operatorname{rank}(A) \geq t$ (QED).

Acknowledgement. I would like to express my gratitude the referee for suggesting the present form of the stated material.

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REZIME

NOTA O STANDARDNOJ MATRIČNOJ REPREZENTACIJI MATROIDA

Osnovni rezultat ovog rada je: Ako je $[I_k, A]$ SMR matroida u odnosu na bazu B, tada je $\operatorname{rank}(A) \geq C(B)$, gde je $C(B)$ maksimalan broj.

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CONNECTEDNESS OF THE NON-COMPLETE EXTENDED p-SUM OF GRAPHS

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ABSTRACT

We extend the definition of the non-complete extended p-sum (NEPS) of graphs to digraphs (digraphs can have multiple arcs and/ or loops). Using the spectral method we prove a theorem giving the necessary and sufficient condition for a NEPS of strongly connected digraphs to be strongly connected. Some related results are obtained.

Let B be a set of n-tuples $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, of symbols 0 and 1, which does not contain an n-tuple $(0,0,\ldots,0)$.

DEFINITION. *The non-complete extended p-sum (NEPS) with a basis* B *of digraphs* $G_1, G_2, \ldots, G_n$ *is the digraph* G *whose set of vertices is the Cartesian product of the sets*

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of vertices of digraphs $G_1, G_2, \ldots, G_n$. *For two vertices* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *construct all the possible arc selections of the following type. For each* $\beta \in B$ *and for any* i $(i = 1, 2, \ldots, n)$ *select an arc from* x_i *to* y_i *in* G_i *if* $\beta_i = 1$ *and suppose* $x_i = y_i$ *if* $\beta_i = 0$. *The number of arcs going from* $(x_1, x_2, \ldots, x_n)$ *to* $(y_1, y_2, \ldots, y_n)$ *is equal to the number of such selections.*

If B consists of all the possible n-tuples (of course, without the n-tuple $(0,0,\ldots,0)$) the operation is called a strong product. The incomplete p-sum (complete p-sum, or briefly, p-sum) is obtained if B consists of (all the possible) n-tuples with exactly p 1's. If $p = n$, the p-sum is called a product.

Some special cases of this definition have already appeared in literature (see, for example [1,p.303], [6], [7]).

Let $A \otimes B$ denote the Kronecker product of matrices A and B. Let $(A)_{xy}$ be the element of the matrix A from the row corresponding to vertex x and the column cerresponding to vertex y of a graph which corresponds to A.

THEOREM 1. *The NEPS* G *with the basis* B *of digraphs* $G_1, G_2, \ldots, G_n$, *whose adjacency matrices are* $A_1, A_2, \ldots, A_n$, *has the following adjacency matrix*

$$A = \sum_{\beta \in B} A_1^{\beta_1} \otimes A_2^{\beta_2} \otimes \ldots \otimes A_n^{\beta_n} .$$

P r o o f. In each of the digraphs $G_1, G_2, \ldots, G_n$ let the vertices be ordered (labelled). We shall order, lexicographically, the vertices of G (which represent the ordered n-tuples of the vertices of digraphs $G_1, G_2, \ldots, G_n$) and form the adjacency matrix A according to this ordering.

By virtue of the properties of the Kronecker product of matrices, the entries of A are

$$(1)\quad (A)_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \sum_{\beta \in B} (A_1^{\beta_1})_{x_1 y_1} \cdots (A_n^{\beta_n})_{x_n y_n} .$$

By virtue of the lexicographic ordering, (1) holds if and only if for each $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in B$ there exist $(A_i)_{x_i y_i}$ arcs leading from x_i to y_i in G_i if $\beta_i = 1$, and $x_i = y_i$ if $\beta_i = 0$.

This completes the proof of the Theorem.

The results in [6] are a special case of this theorem.

THEOREM 2. *For* $i = 1,2,\ldots,n$ *let* G_i *be a digraph with* n_i *vertices, and let* $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$ *be the spectrum of* G_i. *Then the spectrum of NEPS with the basis* B *of digraphs* $G_1, G_2, \ldots, G_n$ *consists of all the possible values of* $\Lambda_{i_1,\ldots,i_n}$, *where*

$$\Lambda_{i_1,\ldots,i_n} = \sum_{\beta \in B} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ni_n}^{\beta_n}, \quad (i_k=1,\ldots,n_k;\ k=1,\ldots,n).$$

The proof coincides with the proof in the case of undirect graphs (c.f.Theorem 2.23 in [3]).

It is obvious that NEPS G is not strongly connected if any one of digraphs $G_1, G_2, \ldots, G_n$ is not strongly connected or if B has not the property (D) that for every $j \in \{1,2,\ldots,n\}$ there exists in B at least one n-tuple $(\beta_1, \beta_2, \ldots, \beta_n)$ with $\beta_j = 1$. (This condition implies that the NEPS, effectively, depends on each G_i.)

Let h be the greatest common divisor of the lengths of all the cycles in a digraph G. The digraph is called primitive if it is strongly connected and $h = 1$ [5,p.210], and inprimitive if it is strongly connected and $h > 1$. In the second case h is called the index of imprimitivity (h is the index of imprimitivity of the adjacency matrix of the digraph G as well [2,p.183]).

THEOREM 3. *Let* $G_1, G_2, \ldots, G_n$ *be strongly connected digraphs each containing at least two vertices. Suppose also that* $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ $(\{i_1, i_2, \ldots, i_s\} \subset \{1,2,\ldots,n\})$ *are*

imprimitive with the imprimitivity indices $h_{i_1}, h_{i_2}, \dots, h_{i_s}$, *respectively. The NEPS with the basis* B *satisfying condition* (D), *of digraphs* $G_1, G_2, \dots, G_n$ *is a strongly connected digraph if and only if for every non-empty subset* $\{j_1, j_2, \dots, j_k\}$ *of* $\{i_1, i_2, \dots, i_s\}$ *and for every choice of integers* $\ell_{j_1}, \ell_{j_2}, \dots, \ell_{j_k}$ $(1 \le \ell_{j_t} \le h_{j_t} - 1;\ t = 1,2,\dots,k)$, *there exists* $\beta \in B$ *such that* $\{j_1, j_2, \dots, j_k\} \cap \{r \mid \beta_r = 1\} = \{\nu_1, \nu_2, \dots, \nu_m\} \neq \emptyset$ *and*

$$\frac{\ell_{\nu_1}}{h_{\nu_1}} + \frac{\ell_{\nu_2}}{h_{\nu_2}} + \dots + \frac{\ell_{\nu_m}}{h_{\nu_m}}$$

is not an integer.

Moreover, the number of strong components of NEPS is equal to the number of solutions in integers x_i $(0 \le x_i \le h_i - 1)$, x_β *of the following system of equations*

$$\frac{x_{i_1}}{h_{i_1}}\beta_{i_1} + \frac{x_{i_2}}{h_{i_2}}\beta_{i_2} + \dots + \frac{x_{i_s}}{h_{i_s}}\beta_{i_s} = x_\beta \quad (\beta \in B).$$

P r o o f. According to Theorem 0.4 and 0.5 from [3] a digraph G, with an adjacency matrix A, is strongly connected if and only if its index r is a simple eigenvalue and if the positive eigenvectors belong to r both in A and A^T. However, if the index r has a multiplicity p, the other conditions being the same, then G has exactly p strong components.

Let $r_1, r_2, \dots, r_n$ be indices of digraphs $G_1, G_2, \dots, G_n$, respectively, and let $u_1, u_2, \dots, u_n$ $(v_1, v_2, \dots, v_n)$ be positive eigenvectors [3,p.18] (Theorem of Frobenius) belonging to $r_1, r_2, \dots, r_n$ in $A_1, A_2, \dots, A_n$ $(A_1^T, A_2^T, \dots, A_n^T)$, respectively. Then, from Theorem 1, it immediately follows that $u = u_1 \otimes u_2 \otimes \dots \otimes u_n$ $(v = v_1 \otimes v_2 \otimes \dots \otimes v_n)$ is the positive eigenvector belonging to the index $\Lambda = \sum_{\beta \in B} r_1^{\beta_1} r_2^{\beta_2} \dots r_n^{\beta_n}$ in $A(A^T)$.

By Theorem 2 the index Λ of NEPS can be obtained only from those eigenvalues of the digraphs G_i $(i = 1,2,\ldots,n)$ which have a modulus equals to r_i. All these eigenvalues of G_j can be written in the form $r_j \exp(\ell_j \frac{2\pi}{h_j})$, $0 \leq \ell_j \leq h_j-1$, $(\exp(t) = e^{ti},\ i^2 = -1)$ (Theorem of Frobenius).

By Theorem 2 we have

$$(2)\quad \Lambda = \sum_{\beta\in B} r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n} \exp\left(\left(\frac{\ell_{i_1}}{h_{i_1}}\beta_{i_1} + \frac{\ell_{i_2}}{h_{i_2}}\beta_{i_2} + \ldots + \frac{\ell_{i_s}}{h_{i_s}}\beta_{i_s}\right)2\pi\right).$$

From (2), it follows that the multiplicity of the index Λ is equal to the number of solutions in integers x_i, $0 \leq x_i \leq h_i-1$ of the system of equations given above. Furthermore, Λ is a simple eigenvalue if for each choise $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_s}$, $0 \leq \ell_{i_t} \leq h_{i_t}-1$ $(t = 1,2,\ldots,s)$ with at least one $\ell_{i_t} > 0$, at least one sumand in Λ is different from $r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n}$ (i.e. the argument of the operator exp is different from $2k\pi$, $k \in Z$).

From these facts, the statement of the theorem follows.

The strong components of NEPS in this theorem are its components also, i.e. there are no arcs between different strong components (Theorem 7′ from [4,p.376]).

The following theorem is a specialization of the preceding one.

THEOREM 4. *Let* $G_1, G_2, \ldots, G_n$ *be strongly connected digraphs each containing at least two vertices and let* $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ $(\{i_1, i_2, \ldots, i_s\} \subset \{1,2,\ldots,n\})$ *be imprimitive with imprimitivity indices* $h_{i_1}, h_{i_2}, \ldots, h_{i_s}$*, respectively. Then the p-sum of* $G_1, G_2, \ldots, G_n$ *is a strongly connected digraph if and only if one of the following condition holds:*

1° $\quad n - s \geq p - 1$;

2° $\quad n - s < p - 1$,

and for every non-empty subset $\{j_1,j_2,\dots,j_k\}$ *of* $\{i_1,i_2,\dots,i_s\}$ $(n-p+2 \le k \le s)$ *and for each choise of integers* $\ell_{j_1},\ell_{j_2},\dots,\ell_{j_k}$; $1 \le \ell_{j_t} \le h_{j_t}-1$ $(t = 1,2,\dots,k)$ *there exists a non-empty subset* $\{\nu_1,\nu_2,\dots,\nu_m\}$ *of* $\{j_1,j_2,\dots,j_k\}$ $(p+k-n \le m \le \min(k,p))$ *such that*

$$\frac{\ell_{\nu_1}}{h_{\nu_1}} + \frac{\ell_{\nu_2}}{h_{\nu_2}} + \dots + \frac{\ell_{\nu_m}}{h_{\nu_m}}$$

is not an integer.

The number of strong components in the p-sum is equal to the number of solution in integers x_i $(0 \le x_i \le h_i-1)$, $x_{j_1j_2\dots j_p}$ *of the following system of equations*

$$\frac{x_{j_1}}{h_{j_1}} + \frac{x_{j_2}}{h_{j_2}} + \dots + \frac{x_{j_p}}{h_{j_p}} = x_{j_1j_2\dots j_p}$$

where $\{j_1,j_2,\dots,j_p\}$ *runs over all p-subsets of* $\{1,2,\dots,n\}$.

For $p = n$, from this theorem, it follows that the product of digraphs $G_1,G_2,\dots,G_n$ is strongly connected if and only if $h_{i_1},h_{i_2},\dots,h_{i_s}$ are the relative prime in pairs (which is well known [7]) and have as many strong components as is the number of solutions in integers x_i $(0 \le x_i \le h_i-1)$, x of the equation

$$\frac{x_{i_1}}{h_{i_1}} + \frac{x_{i_2}}{h_{i_2}} + \dots + \frac{x_{i_s}}{h_{i_s}} = x \,.$$

It can be easily shown that this equation has exactly[1)]

$$\frac{h_{i_1}\cdot h_{i_2}\dots h_{i_s}}{\text{l.c.m.}(h_{i_1},h_{i_2},\dots,h_{i_s})}$$

soluitons, which implies the result from [7].

---

1) *l.c.m. denotes the lowest common multiple*

Finally, we shall prove a simple result concerning regularity properties. A digraph is caled a regular of degree r if each indegree and each outdegree equals r. It is easy to see that a digraph is regular if the eigenvector $(1,1,\ldots,1)$ belongs to its index both in the adjacency matrix and its transpose.

THEOREM 5. *The NEPS of regular digraphs is a regular digraph.*

P r o o f. The vector $u_1 \otimes u_2 \otimes \ldots \otimes u_n$, where $u_1, u_2, \ldots, u_n$ are eigenvectors of indices of $G_1, G_2, \ldots, G_n$, is an eigenvector belonging to the index of NEPS.

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REZIME

POVEZANOST NEPOTPUNE PROŠIRENE p-SUME GRAFOVA

U radu je proširena definicija nepotpune proširene p-sume (NEPS) grafova na digrafove. Korišćenjem spektralnog metoda dokazana je teorema (Theorem 3) koja daje potrebne i dovoljne uslove da NEPS jako povezanih digrafova bude jako povezan digraf.

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GRAPHS WHOSE ENERGY DOES NOT EXCEED 3

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ABSTRACT

In this paper, we determine all the finite connected graphs, whose energy (i.e. the sum of all positive eigenvalues) does not exceed 3. To do this, we consistently apply the method of forbidden subgraphs.

INTRODUCTION

Throughout the paper, we shall consider only finite connected graphs, having no loop or multiple edges. The spectrum of such a graph G is the set of eigenvalues of its 0-1 adjacency matrix $A(G)$. The sum of all its positive eigenvalues is denoted by $S(G)$, and called the energy of G.

For any real $a \geq 1$, we consider the class of graphs

$$P(a) = \{G \mid S(G) \leq a\},$$

and, in this paper, we shall completely describe the class $P(3)$.

Briefly, any graph $G \in P(3)$ is here called - admissible, and any other graph - impossible.

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Key words and phrases: Finite connected graphs, the energy.

Let next G' be any connected (induced) subgraph of a graph G, which is denoted by $G' \subseteq G$. Since by the known interlacing theorem [1, p.19] $S(G') \leq S(G)$, we have that any connected subgraph of an admissible graph is admissible, too. This implies that the method of forbidden subgraphs can be consistently applied.

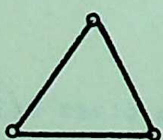
Throughout the paper, K_n, P_n, C_n will be the complete graph, the path and the cycle with $\underline{n}$ vertices, respectively, while $K_{n,m}$ is the complete bipartite graph with $n+m$ vertices.

In this paper, without special mention, we shall often use the lists of spectra of all connected graphs with 2, 3, 4 or 5 vertices (see [1]), or connected graphs with 6 vertices (112 graphs; an internal publication). So, using these lists, for each particular graph with this number of vertices, we shall determine whether it is admissible or not.

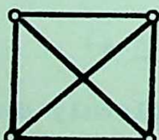
RESULTS

Denote by a circle any set of isolated vertices, and by the line between two circles the fact that there are all edges between these circles.

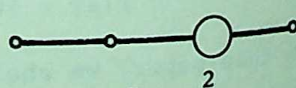
Then, by the direct inspection of the spectra of all the connected graphs with 2, 3, 4, 5 or 6 vertices, we have that all the admissible graphs with at most 6 vertices, belong to one of the following classes of graphs:



K_3



K_4



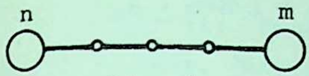
R

$(n,m \geq 1)$

K(n,m)

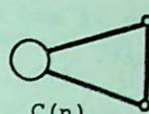
$(n,m \geq 1)$

A(n,m)



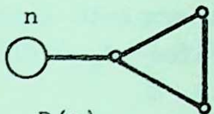
$(n,m \geq 1)$

B(n,m)



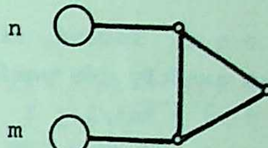
$(n \geq 2)$

C(n)



$(n \geq 1)$

D(n)



$(n,m \geq 1)$

E(n,m)

Now, we shall determine the exact values of parameters for which the above graphs are admissible.

LEMMA 1. *The graph* $K(n,m)$ $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1,\ m \leq 9$;

2) $n = 2,\ m = 2,3,4$;

3) $n = m = 3$.

P r o o f. As is easily seen, this graph is admissible if and only if $nm \leq 9$ holds, whence the statement is immediate. □

LEMMA 2. *The graph* $A(n,m)$ $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1,\ m = 1,2,3$;

2) $n = m = 2$.

P r o o f. Immediately, this graph is admissible if and only if $\sqrt{n} + \sqrt{m} \leq 2\sqrt{2}$, whence the statement is obvious. □

LEMMA 3. *The graph* $B(n,m)$ $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1,\ m = 1,2,3$;

2) $n = m = 2$.

P r o o f. As is easily seen, the graph $B(n,m)$ is an admissible graph if and only if $n + m + 2\sqrt{n+m} \leq 9$, whence the statement. □

LEMMA 4. *The graph* $C(n)$ $(n \geq 2)$ *is admissible iff* $n = 2,3$.

P r o o f. Indeed, since it is a complete 3-partite graph, it has exactly one positive eigenvalue $r_n = r(C(n))$, and $r_n = (1 + \sqrt{1 + 8n})/2 \leq 3$ iff $n = 2,3$. □

LEMMA 5. *The graph* $D(n)$ $(n \geq 1)$ *is admissible iff* $n = 1,2$.

P r o o f. The graphs $D(1)$, $D(2)$ are admissible, and $D(3)$ is not so. Whence, all $D(n)$ $(n \geq 3)$ are non-admissible, also. □

LEMMA 6. *The graph* $E(n,m)$ $(n,m \geq 1)$ *is admissible iff* $n = m = 1$.

P r o o f. Indeed, since $E(1,1)$ is admissible, and $E(1,2)$ is an impossible graph, we have that $E(n,m)$ for $n \geq 2$ or $m \geq 2$, are impossible graphs. □

Now, we shall prove the main result of the paper.

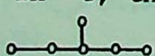
THEOREM 1. *Each admissible graph* G *is one of the graphs displayed in Figure 1.*

P r o o f. We distinguish the next three cases:

I. There is no C_3 or C_4 in G as a subgraph;

II. There is a C_3 in G;

III. There is C_4 but no C_3 as an induced subgraph in G.

CASE I. Since C_n $(n \geq 5)$ cannot be a subgraph of an admissible G, we conclude that, in this case, there is no contour in G; thus G is a tree. Since, next, there is no P_5 or  in G, we have that G must be one of the

graphs $K(1,n)$ $(n \geq 1)$, $A(n,m)$ $(n,m \geq 1)$, $B(n,m)$ $(n,m \geq 1)$.

CASE II: Denote the vertices of the $L = C_3$ in G by 1,2,3. Next, denote by T_i $(i = 1,2,3)$ the vertices of G which are (with respect to L) adjacent exactly to the vertex $\underline{i}$; The denotations T_{12}, T_{13}, T_{23} and T_{123} have a similar meaning. Put

$$T = T_1 + T_2 + T_3 + T_{12} + T_{13} + T_{23} + T_{123} .$$

Next, denote by $\tilde{T}_i$ the vertices of G (non-adjacent to L), which are (with respect to T), adjacent exactly to some vertices of T_i; the denotations $\tilde{T}_{ij}$ $(i \neq j)$ and $\tilde{T}_{123}$ have a similar meaning.

Now, we are interested to determine the edge structure of each particular subset between T_i, T_{ij} and T_{123}, as well as the edge structure between these subsets.

For any two subsets A,B, we use the denotation $A/A = 0$ if A consists of the isolated vertices only, $A/A = 1$ if it is complete, $A/B = 0$ or 1 or $\emptyset$ or $*$, if there is no edge between A and B, or there are all such edges, or A and B are not consistent, or we cannot determine this structure, respectively.

All the above information is obtained by choosing two arbitrary vertices $a \in A$, $b \in B$, then testing the subgraph $123ab$ in the two possible cases - whether a,b are adjacent or not.

So, by the impossible graphs or order 5, we obtain the following relations easily:

$$T_i/T_i = 0 \quad (i = 1,2,3), \qquad T_{ij}/T_{ij} = 0 \quad (i \neq j),$$

$$|T_{123}| \leq 1, \quad T_i/T_j = 0 \quad (i \neq j), \quad T_i/T_{ij} = \emptyset,$$

$$T_i/T_{jk} = \emptyset, \quad T_i/T_{123} = \emptyset, \quad T_{ij}/T_{ik} = \emptyset,$$

$$T_{ij}/T_{123} = \emptyset .$$

Next, testing the graph 123abc ($a \in T_1$, $b \in T_2$, $c \in T_3$), we also obtain that the 3-tuple T_1, T_2, T_3 is not consistent in G.

Similarly, we obtain that

$$\tilde{T}_i = \emptyset \ , \quad \tilde{T}_{ij} = \emptyset \ , \quad \tilde{T}_{123} = \emptyset \ ,$$

whence follows that each admissible G, in case II, consists of $L = C_3$ and eventually of the classes T_i, T_{ij}, T_{123}.

In view of all the above results, excluding the symmetric cases, we have that G, in case II, consists of one of the following subsets: only $L = C_3$, $L + T_1$, $L + T_{12}$, $L + T_{123}$, $L + T_1 + T_2$.

Consequently, G is one of the following graphs: C_3, K_4, $C(n)$ $(n \geq 2)$, $D(n)$ $(n \geq 1)$, $E(n,m)$ $(n,m \geq 1)$.

CASE III. Denote the vertices of $L = C_4$ in G by 1,2,3,4.

Then, similarly as in the previous case, we have the subsets T_i, T_{ij}, T_{ijk} and T_{1234} in G.

By assumption, or by forbidden subgraphs, we conclude easily that

$$T_{12} = T_{23} = T_{34} = T_{14} = \emptyset, \quad T_{ijk} = \emptyset \ \text{ and } \ T_{1234} = \emptyset.$$

so in T there remain only the subsets T_i $(i = 1,2,3,4)$ and T_{13}, T_{24}.

By the impossible graphs of order 6 (and by the assumption), we conclude that $\tilde{T}_i = \emptyset$ and $\tilde{T}_{13} = \tilde{T}_{24} = \emptyset$, thus, in this case, each admissible G consists only of the subsets L, T_1, T_2, T_3, T_4 T_{13} and T_{24}.

As in case II, we conclude that

$$|T_i| \leq 1, \qquad T_{13}/T_{13} = T_{24}/T_{24} = 0 \ ,$$

$$T_i/T_j = \emptyset, \quad T_i/T_{13} = T_i/T_{24} = \emptyset, \quad T_{13}/T_{24} = 1.$$

Hence, excluding the symmetric cases, we have that G consists of one of the following subsets: $L = C_4$, $L + T_1$, $L + T_{13}$, $L + T_{13} + T_{24}$.

Consequently, in case III, G must be one of the following graphs: $K(2,2)$, R, $K(2,n)$ $(n \geq 3)$, $K(n,m)$ $(n,m \geq 3)$, which completes the proof. □

Note that Theorem 1 and Lemma 1-6 describe the class $P(3)$ completely.

Note, still, that the previous results imply that class $P(a)$ is finite if $a = 3$. In the following theorem, we shall prove this for any $a \geq 1$.

THEOREM 2. *The class* $P(a)$ *is finite, for any* $a \geq 1$.

Proof. Choose an arbitrary graph $G \in P(a)$ and any (not necessarily induced) subgraph $K(1,n)$. Then, since $a \geq S(G) \geq r(G) \geq \sqrt{n} = r(K(1,n))$, where $r(G)$ is the spectral radius of G (see Theorem 0.9 [1, p.15] for the last inequality), we conclude that $K(1,n) \in P(a)$, thus that all such n's are uniformly bounded by $b = a^2$. Hence, the degrees of all the vertices in G cannot exceed the constant b.

Next, choose any path P_n. Since, for an arbitrary $q \in N$, its q-th positive eigenvalue tends to 2 as $n \to \infty$, we get that $S(P_n) \to \infty$ as $n \to \infty$. Consequently, for any path $P_n \in P(a)$ we have that all n's are uniformly bounded by a constant $k = f(a)$.

Now, assume, contrarily to the statement, that the set $P(a)$ is finite for an $a \geq 1$. Then, it is seen easily that either there is a sequence of the complete bipartite graphs $K(1,n_i) \in P(a)$ $(n_1 < n_2 < \ldots)$, or there is a sequence of paths $P(n_i)$ $(n_1 < n_2 < \ldots)$, and both these cases give the contradictions.

This proves the theorem. □

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REZIME

GRAFOVI ČIJA ENERGIJA NE PRELAZI 3

U ovom radu su odredjeni svi konačni povezani grafovi, čija energija (zbir svih pozitivnih svojstvenih vrednosti) ne prelazi 3.

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A COUNTERFEIT COINS PROBLEM

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ABSTRACT

We consider the problem of ascertaining the minimum number of weighings which suffice to determine all counterfeit (heavier) coins in a set of n coins of the same appearance, given a balance scale and the information that there are exactly three heavier coins present. A procedure which is either optimal or suboptimal is constructed for infinitely many n's, i.e., for all $n = 3^k$ $(k=1,2,3,\dots)$.

1. INTRODUCTION

Consider the following problem. Let $X = \{c_1, c_2, \dots, c_n\}$ be a set of n coins indistinguishable except that exactly m $(m \leq n)$ of them are slightly heavier than the rest (in the sense specified below). Given a balance scale, we want to find an optimal weighing procedure, i.e., a procedure which minimizes the maximum number of steps (weighings) which are required to identify all heavier coins. For some discussion of these matters in greater detail, see [1], [2], [3], [4] and [5].

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We suppose that all heavier coins are of equal weight, and so are all light coins. If λ is the weight of a light (good) coin, then the weight of a heavy (counterfeit) coin is less than $\frac{m+1}{m}\lambda$, so that the larger of the two numerically unequal subsets of X is always the heavier. This means that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of two subsets of X is heavier but not by how much.

Consider a pair (A,B) of numerically equal disjoint subsets of X. Step (A,B) will mean the balancing of A against B. The following outcomes are possible:

(a) The sets balance, symbolized by $A = B$,

(b) The sets do not balance, symbolized by $A \neq B$. We use the notation, if necessary, $A > B$, $A < B$, where $>$ and $<$ between two sets means "is heavier than" and "is lighter than" respectively.

Let $P_n^m(\ell)$ denote any procedure which enables us to identify all heavier coins, if there are exactly m of them in the set of n coins, ℓ being the maximum number of weighings to be required. $P_n^m(\leq \ell)$ will mean a procedure for which the maximum number of steps to be required is not greater than ℓ. A procedure $P_n^m(\ell)$ is said to be optimal if no one procedure $P_n^m(r)$ exists for some $r < \ell$. We write $\mu_m(n) = \ell$ if there is an optimal procedure $P_n^m(\ell)$. A procedure is said to be suboptimal if $\mu_m(n) = \ell-1$. It follows by information-theoretical reasonings that

$$\mu_m(n) \geq \lceil \log_3 \binom{n}{m} \rceil$$

where $\lceil x \rceil$ denotes the least integer $\geq x$. It is well known that $\mu_1(n) = \lceil \log_3 n \rceil$. In [6] it is proved that

$$\lceil \log_3 \binom{n}{2} \rceil \leq \mu_2(n) \leq 1 + \lceil \log_3 \binom{n}{2} \rceil$$

and a corresponding procedure is construced such that the lower bound is reached for an infinite set of n´s.

In this paper, a procedure for three counterfeit coins problem is constructed, which is either optimal or suboptimal for infinitely many n's, i.e., for all $n = 3^k$ ($k=1,2,3,\ldots$).

2. RESULTS

THEOREM. *If* $n = 3^k$ ($k=1,2,3,\ldots$), *then*

$$\lceil \log_3 \binom{n}{3} \rceil \le \mu_3(n) \le 1 + \lceil \log_3 \binom{n}{3} \rceil .$$

P r o o f. It is easy to check that $3^{3k-2} < \binom{3^k}{3} < 3^{3k-1}$, i.e., $\lceil \log_3 \binom{3^k}{3} \rceil = 3k-1$, for $k \ge 2$. Now, the statement will be proved by the inductive construction of a procedure $P^3_{3^k}$ ($\le 3k$), for $k \ge 1$.

For $k = 1$, we have the trivial (empty) strategy $P^3_3(0)$ which satisfies the statement.

Suppose that a procedure $P^3_{3^k}$ ($\le 3k$) is constructed. Then, a procedure $P^3_{3^{k+1}}$ ($\le 3k+3$) can be constructed as follows.

Let $A = \{c_1, \ldots, c_{3^k}\}$, $B = \{c_{3^k+1}, \ldots, c_{2\cdot 3^k}\}$, $C = \{c_{2\cdot 3^k+1}, \ldots, c_{3^{k+1}}\}$.

Step 1. (A,B).

Step 2. (B,C).

It is sufficient to analyse four cases ((a)-(d) below); any other possible case is quite analogous to one of these four.

(a) $A = B$, $B = C$. It is clear that each of the sets A,B,C, contains exactly one heavier coin. We continue by successive applications of the procedure $P^1_{3^k}(k)$, three times, to the sets A,B and C independently. It follows that all heavier coins will be found after 3k+2 steps.

(b) $A = B$, $B < C$. Now, all heavier coins are in the set C.

We apply a procedure $P^3_{3^k}$ ($\leq 3k$), which can be constructed by the induction hypothesis, to the set C. All heavier coins will be found after at most 3k+2 steps.

(c) $A < B$, $B < C$. We conclude that one heavier coin is in the set B and two of them are in the set C. They all can be found by applying two independent procedures, $P^1_{3^k}(k)$ and $P^2_{3^k}(2k)$, to the sets B and C respectively. The construction of a procedure $P^2_{3^k}(2k)$ is given in [6]. So, all heavier coins will be found after 3k+2 steps.

(d) $A < B$, $B > C$. Go to Step 3.

Step 3. (A,C).

There are three possible cases.

(da) $A = C$. We conclude that all heavier coins are in the set B, and continue by the application of a procedure $P^3_{3^k}(\leq 3k)$ to the set B. All heavier coins will be found after at most 3k+3 steps.

(db) $A < C$. Now, one heavier coin is in the set C and two of them are in the set B. We continue similarly as in the case (c), by applying two procedure, $P^1_{3^k}(k)$ and $P^2_{3^k}(2k)$, to the sets C and B respectively. All the heavier coins will be found after 3k+3 steps.

(dc) $A > C$. This case is quite similar to the case (db); now, one heavier coin is in the set A and two of them are in the set B, so, 3k+3 steps will suffice.

A procedure $P^3_{3^{k+1}}$ ($\leq 3k+3$) is constructed and the theorem is proved.

REMARK. It is easy to see that, for $k \geq 3$, the constructed procedure is in fact a $P^3_{3^k}(3k)$ procedure; only for $k = 1$ and $k = 2$, it is a $P^3_{3^k}(< 3k)$ procedure. For $k = 1$, we have the trivial procedure $P^3_3(0)$; for $k = 2$, $P^2_{3^k}(2k)$ which is used

in (db) and (dc) become $P_3^2(2)$ and can be replaced by a $P_3^2(1)$ procedure, while $P^3_{3^k}(\leq 3k)$ used in (da) become $P_3^3(0)$. So, for $k = 2$, we obtain a procedure $P_9^3(5)$, which is optimal since the information-theoretical lower bound is reached.

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REZIME

JEDAN PROBLEM O NEISPRAVNIM NOVČIĆIMA

Posmatra se problem odredjivanja minimalnog broja merenja dovoljnih za identifikaciju svih neispravnih (težih) novčića u skupu od n novčića, uz pretpostavku da se u tom skupu nalaze tačno tri neispravna novčića. Jedna procedura koja je optimalna ili suboptimalna, konstruisana je za jedan beskonačan skup vrednosti parametra n.

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ПЕРЕЧИСЛЕНИЕ МОНОТОННЫХ СИММЕТРИЧЕСКИХ ФУНКЦИЙ ТРЕХЗНАЧНОЙ ЛОГИКИ

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РЕЗЮМЕ

В работе доказано что число n-местных монотонных симметрических функций трехзначной логики равно $\binom{2n+3}{n+1}$.

1. ВВЕДЕНИЕ

Пусть $E_3 = \{0,1,2\}$ и P_3^n множество всех функций $f : E_3^n \to E_3$.

ОПРЕДЕЛЕНИЕ 1. Функция $f(x_1,\ldots,x_n) \in P_3^n$ называется монотонной относительно порядка $0 < 1 < 2$, если для любых наборов $a = (a_1,\ldots,a_n)$ и $b = (b_1,\ldots,b_n)$, таких что $a \le b$, имеет место соотношение $f(a) \le f(b)$, где $a \le b$ если $a_i \le b_i$ $(1 \le i \le n)$.

Обозначим через M множество всех монотонных функций трехзначной логики.

ОПРЕДЕЛЕНИЕ 2. Функция $f(x_1,\ldots,x_n) \in P_3^n$ называется симметрической если $f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$ для любой перестановки $(y_1,\ldots,y_n)$ переменных $(x_1,\ldots,x_n)$.

В настоящей работе определяется число симметрических n-местных функций трехзначной логики.

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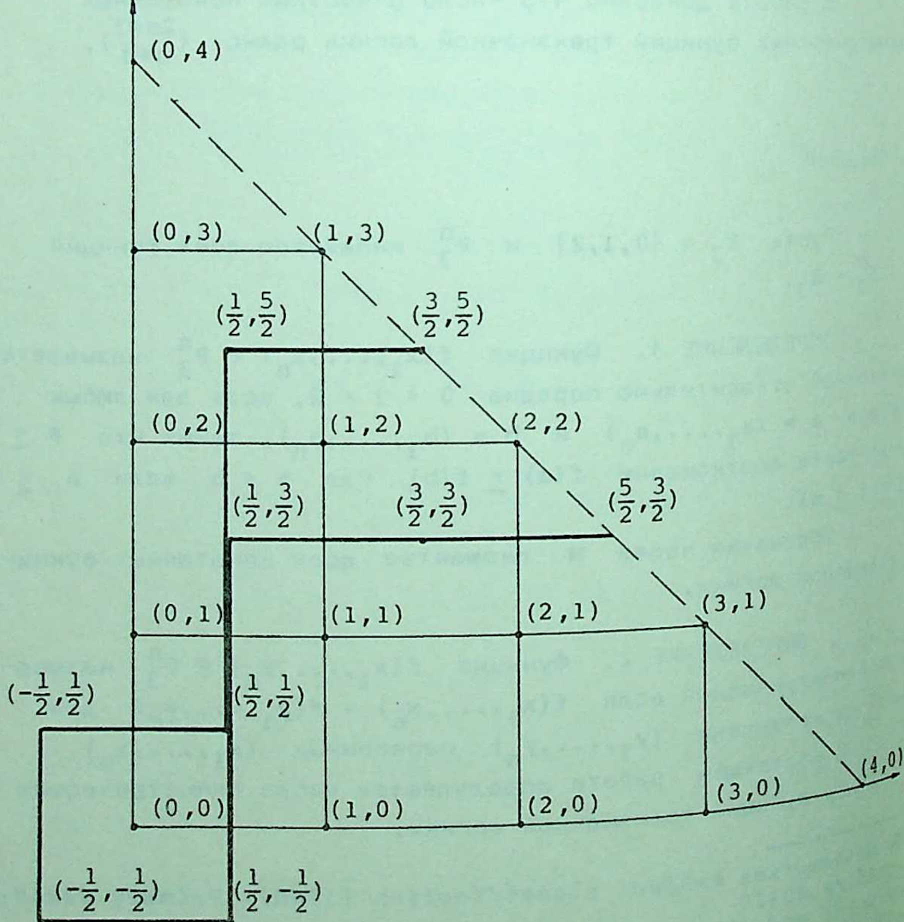
Пусть S_n обозначает множество всех симметрических n-местных функций трехзначной логики и $k(X)$ кардинальное число множества X.

2. ТЕОРЕМА

ТЕОРЕМА. $K(M \cap S_n) = \binom{2n+3}{n+1}$.

Доказательство. Изобразим все наборы $(0,\ldots,0,1,\ldots,1,2,\ldots,2)$ в координатной плоскости точками (x,y). Число нулей в наборе обозначим через x, число двоек через y (тогда число единиц $z = n - x - y$). На рисунке представлени наборы для $n = 4$.



Пусть $L_{n+1}^{(2)}$ множество всех точек $(p,q) \in N$ в координатной плоскости таких что $p \geq 0$, $q \geq 0$, $p + q \leq n$. Легко увидеть что каждая монотонная симметрическая функция $f : E_3^n \to E_3$ однозначно определяет монотонную функцию $F : L_{n+1}^{(2)} \to E_3$, т.е. такую что $(x',y') \leq (x'',y'') \Rightarrow F(x',y') \leq F(x'',y'')$, где $(x',y') \leq (x'',y'') \Leftrightarrow x' \geq x'' \wedge y' \leq y''$. Поэтому $k(M \cap S_n)$ равно числу монотонних функций $F : L_{n+1}^{(2)} \to E_3$.

Пусть A_i $(i = 0,1,2)$ множество точек $(x,y) \in L_{n+1}^{(2)}$ для которих $F(x,y) = i$. Этими множествами функция F определена однозначно.

Для монотонной функции F, множества A_i можно выделит двумя ломаными линиями с началом в точке $(-\frac{1}{2},-\frac{1}{2})$, састоящим от $n+1$ отрезков. Каждий отрезан соединяет точку (u,v) с одной из точек $(u,v+1)$ или $(u+1,v)$. Отрезок, концы которого (u,v) и $(u,v+1)$, обозначим через 1, а отрезок, концы которого точки (u,v) и $(u+1,v)$ - через 0. На рисунке, например, двумя ломаными представлены последовательности 10110 и 01100.

Две последовательности $a_1a_2\ldots a_{n+1}$ и $b_1b_2\ldots b_{n+1}$; $a_i,b_i \in \{0,1\}$, $1 \leq i \leq n+1$, определьют множества A_o,A_1,A_2 монотонной функции тогда и только тогда, когда для каждого k $(1 \leq k \leq n+1)$ в последовательности $a_1a_2\ldots a_k$ число единиц не меньше числа единиц в последовательности $b_1b_2\ldots b_k$. Это имеет место тогда и только тогда, когда в последовательности $c_1c_2\ldots c_{n+1}$, где $c_i = 2a_i + b_i$, для каждого $k \in \{1,2,\ldots,n+1\}$, число двоек в последовательности $c_1c_2\ldots c_k$ не меньше числа единиц.

Прежде чем продолжить доказательство, дадим следующие определения:

ОПРЕДЕЛЕНИЕ 3. Последовательность $c_1c_2\ldots c_n$, $(c_i \in \{0,1,2,3\},\ 1 \leq i \leq n)$ называется <u>характеристической</u> если для каждого k, $1 \leq k \leq n$, число двоек в последовательности $c_1c_2\ldots c_k$ не меньше числа единиц в той же последовательности.

ОПРЕДЕЛЕНИЕ 4. Разность числа двоек и числа единиц в последовательности $c_1c_2\ldots c_n$ называется <u>m-число</u> этой последовательности.

Ясно что последовательность $c_1c_2\ldots c_{n+1}$ определяет множества A_i $(i = 0,1,2)$ некоторой монотонной функции $F : L_{n+1}^{(2)} \to E_3$ тогда и только тогда, когда эта последовательность является характеристической.

Из этого утверждения вытекает следующая лемма:

ЛЕММА 1. Число монотонных симметрических n-местных функций трехзначной логики равно числу характеристических последовательностей $c_1c_2\ldots c_{n+1}$.

Пусть $t(n)$ обозначает число характеристических последовательностей $c_1c_2\ldots c_n$. Из лемми 1, следует что $k(M \cap S_n) = t(n+1)$.

Пусть $t_i(n)$ обзначает число характеристических последовательностей са m-числом i. Из определения характеристической последовательности следует что

$$t_{-1}(n+1) = t_{n+1}(n) = t_{n+2}(2) = 0.$$

ЛЕММА 2. $t_i(n+1) = t_{i-1}(n) + 2t_i(n) + t_{i+1}(n)$, $0 \leq i \leq n+1$.

Доказательство. Вытекает из следующего утверждения:

m-число характеристической последовательности $c_1c_2\ldots c_{n+1}$ равно i тогда и только тогда, когда выполнено одно из следующих трех условий:

(1) m-число характеристической последовательности $c_1c_2\ldots c_n$ равно $i-1$ и $c_{n+1} = 2$;

(2) m-число характеристической последовательности $c_1c_2\ldots c_n$ равно i и $c_{n+1} = 0$ или $c_{n+1} = 3$;

(3) m-число характеристической последовательности $c_1c_2\ldots c_n$ равно $i+1$ и $c_{n+1} = 1$.

ЛЕММА 3. $t_i(n) = \binom{2n}{n+i} - \binom{2n}{n+i+2}$, $0 \le i \le n$.

Доказазельство. Для $n = 1$, легко проверяется что $t_o(1) = 2$ (последователности 0 и 3), $t_1(1) = 1$ (последовательность 2).

Пусть утверждение теоремы верно для $n = k$. Докажем, что тогда оно верно и для $n = k+1$.

Из лемми 2 следует:

$$t_i(k+1) = t_{i-1}(k) + 2t_i(k) + t_{i+1}(k) =$$
$$= \binom{2k}{k+i-1} - \binom{2k}{k+i-1+2} + 2\left(\binom{2k}{k+i} - \binom{2k}{k+i+2}\right) +$$
$$+ \binom{2k}{k+i+1} - \binom{2k}{k+i+1+2} = \binom{2k}{k+i+1} + 2\binom{2k}{k+i} + \binom{2k}{k+i-1} -$$
$$- \binom{2k}{k-i-3} - 2\binom{2k}{k+i+2} - \binom{2k}{k+i+1} = \binom{2k+1}{k+i+1} + \binom{2k+1}{k+i} -$$
$$- \binom{2k+1}{k+i+3} - \binom{2k+1}{k+i+2} = \binom{2k+2}{k+i+1} - \binom{2k+2}{k+i+3} .$$

Лемма доказана.

ЛЕММА 4. Число характеристических последовательностей $c_1c_2\ldots c_n$ равно $\binom{2n+1}{n}$.

Доказательство.

$$t(n) = \sum_{i=0}^{n} t_i(n) = \sum_{i=0}^{n} \left(\binom{2n}{n+i} - \binom{2n}{n+i+2}\right) =$$
$$= \binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n}.$$

Из лемм 1 и 4 непосредственно следует утверждение теоремы.

3. НЕКОТОРЫЕ СЛЕДСТВИЯ ТЕОРЕМЫ

Следствие 1. $\sum_{r=0}^{n} 2^{n-r} \binom{n}{r}\binom{r}{\lfloor \frac{r}{2} \rfloor} = \binom{2n+1}{n}$.

($\lfloor x \rfloor$ обозначает наибольшее целое число, не превосходящее x.) Следствие доказывается пользуясь двумя следующими леммами:

ЛЕММА 5. ([1]) Существует $\binom{r}{i} - \binom{r}{i-1}$ характеристических последовательностей $c_1c_2\ldots c_r$ таких, что $c_j \in \{1,2\}$, $1 \le j \le r$ и что число единиц в последовательности равно i.

ЛЕММА 6. Существует $\binom{r}{\lfloor \frac{r}{2} \rfloor}$ характеристических последовательностей $c_1c_2\ldots c_r$ таких, что $c_j \in \{1,2\}$, $1 \le j < r$.

Доказательство. Следует из равенства

$$\sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \left(\binom{r}{i} - \binom{r}{i-1}\right) = \binom{r}{\lfloor \frac{r}{2} \rfloor} .$$

Следствие 1 доказывается вычислением числа харантерических последовательностей $c_1c_2\ldots c_n$, в которых имеется r единиц и двоек, суммированием по r и пользованием леммы 4. В статьи [3] доказывается следствие 1 другим образом и определяется число $k(M \cap S_n)$ пользуясь этим следствием.

Следствие 2.

$$\sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} 2^{n-2i-k}\binom{n}{2i+k}\left(\binom{2i+k}{i} - \binom{2i+k}{i-1}\right) = \binom{2n}{n+k} - \binom{2n}{n+k+2}.$$

Доказательство. Вытекает из лемм 3 и 5.

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REZIME

PREBROJAVANJE MONOTONIH SIMETRIČNIH FUNKCIJA TROZNAČNE LOGIKE

U radu je dokazano da je broj n-arnih simetričnih monotonih funkcija troznačne logike

$$\binom{2n+3}{n+1} .$$

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ПЕРЕЧИСЛЕНИЕ СИММЕТРИЧЕСКИХ ФУНКЦИЙ ПРЕДПОЛНЫХ КЛАССОВ ТРЕХЗНАЧНОЙ ЛОГИКИ

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РЕЗЮМЕ

Целью настоящей работы является перечисление симметрических функций предполных классов трехзначной логики.

1. ВВЕДЕНИЕ

Пусть P_k обозначает множество всех функций $f(x_1,\dots,x_n)$, аргументы которых определены на множестве $E^k = \{0,1,\dots,k-1\}$, и таких, что $f(\alpha_1,\dots,\alpha_n) \in E^k$, когда $\alpha_i \in E^k$ $(i = 1,\dots,n)$.

Мы говорим, что функция $f(x_1,\dots,x_i,\dots,x_n)$ существенно зависит от аргумента x_i если найдутся два набора $\alpha = (\alpha_1,\dots,\alpha_{i-1},\alpha_i,\alpha_{i+1},\dots,\alpha_n)$, $\alpha' = (\alpha_1,\dots,\alpha_{i-1},\alpha_i',\alpha_{i+1},\dots,\alpha_n)$, где $\alpha_i \neq \alpha_i'$, таких что $f(\alpha) \neq f(\alpha')$.
Все аргументы, от которых функция $f(x_1,\dots,x_n)$ существенно не зависит, называются фиктивными.
Функция является вырожденной, если имеет фиктивные аргументы.

ОПРЕДЕЛЕНИЕ 1. Суперпозицией функций системы $\{f_1,f_2,\dots,f_s,\dots\} \subset P_k$ называется:

a) любая функция, которая получается из функций системы путем замены переменных.

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б) любая функция, которая получается путем замены переменных и добавления любого конечного числа фиктивных аргументов из функций

$F(F_1(y_{11},\dots,y_{1m_1}),\dots,F_p(y_{p1},\dots,y_{pm_p}))$, где $F(y_1,\dots,y_p)$ - суперпозиция функций системы и либо $F_i(y_{i1},\dots,y_{im_i})$ является суперпозицией функций системы, либо $F_i(y_{i1},\dots,y_{im_i})$ есть y_i $(i = 1,2,\dots,p)$ (здесь не предполагается, что все функции $F, F_1,\dots,F_p$ различные).

ОПРЕДЕЛЕНИЕ 2. Система функций из P_k называется полной в P_k, если каждая функция из P_k является суперпозицией функций этой системы.

ОПРЕДЕЛЕНИЕ 3. Класс функций $N \subseteq P_k$ называется предполным в P_k если N представляет неполную в P_k систему, но присоединение любой функции $f \subseteq P_k \setminus N$ обращает N в полную в P_k систему.

ТЕОРЕМА 1. В P_3 существует в точности следующих 18 предполных классов ([6]):

1. Класс L линейных функций. $f(x_1,\dots,x_n) \in L \iff f(x_1,\dots,x_n) = a_0 + a_1x_1 + \dots + a_nx_n \pmod 3$, $a_j \in \{0,1,2\}$, $0 \le i \le n$.
2. Класс V самодвойственных функций.
   $f(x_1,\dots,x_n) \in V \iff f(x_1 + 1,\dots,x_n + 1) = f(x_1,\dots,x_n)+1.$
3. Класс функций T. Этот класс содержит те и только те функции, которые либо являются функциями, существенно зависящими от одного переменного, либо функциями, выпускающими хоть одно значение.
4. Класс функций T_0.
5. Класс функций T_1.
6. Класс функций T_2.

$f(x_1,\dots,x_n) \in T_i \iff f(i,\dots,i) = i,\ i \in \{0,1,2\}.$

7. Класс функций T_{01}.

8. Класс функций T_{02}.

9. Класс функций T_{12}.

$f(x_1,\ldots,x_n) \in T_{ij}$ тогда и только тогда, если для любого набора $\alpha = (\alpha_1,\ldots,\alpha_n)$, состоящего из i и j, $f(\alpha)$ равно либо i, либо j. ($i,j \in \{0,1,2\}$, $i \neq j$).

10. Класс функций U_{01}.

11. Класс функций U_{02}.

12. Класс функций U_{12}.

Пусть $\{i,j,m\} = \{0,1,2\}$. $f(x_1,\ldots,x_n) \in U_{ij}$ тогда и только тогда, если для любых чисел $1 \leq p_1 < p_2 < \ldots < p_s \leq n$ на всех наборах $(\beta_1,\ldots,\beta_n)$, где

$$\beta_r = \begin{cases} m & \text{при } r = p_\ell \ (\ell = 1,2,\ldots,s) \\ \neq m & \text{в остальных случаях,} \end{cases}$$

функция $f(x_1,\ldots,x_n)$ либо принимает только значения i и j, либо $\equiv m$.

13. Класс M_1 монотонных функций относительно порядка $0 < 1 < 2$. Функция $f(x_1,\ldots,x_n)$ является монотонной, если из $x_i \leq y_i$, $1 \leq i \leq n$ следует $f(x_1,\ldots,x_n) \leq f(y_1,\ldots,y_n)$ для любых наборов $(x_1,\ldots,x_n)$ и $(y_1,\ldots,y_n)$.

14. Класс M_2 монотонных функций относительно порядка $1 < 2 < 0$,

15. Класс M_0 монотонных функций относительно порядка $2 < 0 < 1$.

16. Класс функций B_0

17. Класс функций B_1

18. Класс функций B_2

Пусть $\{i,j,m\} = \{0,1,2\}$. $f(x_1,\ldots,x_n) \in B_i$ тогда и только тогда, если для любых чисел $\alpha_1,\ldots,\alpha_n$ равных j и m, функция $f(x_1,\ldots,x_n)$ на всех наборах $(\beta_1,\ldots,\beta_n)$ таких, что $\beta_s \neq \alpha_s$ $(s = 1,\ldots,n)$, принимает значения, не равные $c(\alpha_1,\ldots,\alpha_n)$, где $c(\alpha_1,\ldots,\alpha_n) = j$ или m.

2. ПЕРЕЧИСЛЕНИЕ СИММЕТРИЧЕСКИХ ФУНКЦИЙ В P_k

ОПРЕДЕЛЕНИЕ 4. n-местная функция $f(x_1,\ldots,x_n) \in P_k$ называется симметрической если $f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$, где $(y_1,\ldots,y_n)$ - любая перестановка $(x_1,\ldots,x_n)$.

Симметрические функции находят приложения в теории контактных схем ([1], [2], [3]).

Из определения вытекает, что значение симметрической n-местной функции одинаково для всех наборов, имеющих одинаковое число 0, одинаковое число 1,..., одинаковое число $k-1$. Поэтому можно писать

$$f[m_0,m_1,\ldots,m_{k-1}] = f(\underbrace{0,\ldots,0}_{m_0},\underbrace{1,\ldots,1}_{m_1},\ldots,\underbrace{k-1,\ldots,k-1}_{m_{k-1}}),\text{ где}$$

$$m_0 + m_1 + \ldots + m_{k-1} = n.$$

Пусть $k(X)_s(n)$ обозначает число симметрических n-местных функций множества X.

ТЕОРЕМА 2. $k(P_k)_s(n) = k^{\binom{n+k-1}{k-1}}$ $(n \geq 0)$.

Доказательство: Число различных наборов $(m_0,\ldots,m_{k-1})$ таких, что $m_0 + \ldots + m_{k-1} = n$ равно числу сочетаний с повторениями n элементов класса k. Каждый набор может иметь k значений $(0,1,\ldots,k-1)$. Из этих фактов вытекает результат.

ТЕОРЕМА 3. Вырожденными n-местными симметрическими функциями в P_k являются только константы $0,1,\ldots,k-1$.

Доказательство: Пусть x_1 фиктивный аргумент функции $f(x_1,\ldots,x_n)$, т.е.

$$f(0,x_2,\ldots,x_n) = f(1,x_2,\ldots,x_n) = \ldots = f(k-1,x_2,\ldots,x_n).$$

Из этого равенства и определения симметрической функции вытекает

$$f(\underbrace{0,\ldots,0}_{m_0},\underbrace{1,\ldots,1}_{m_1},\ldots) = f(1,\underbrace{0,\ldots,0}_{m_0},\underbrace{1,\ldots,1}_{m_1-1},\ldots) =$$

$$= f(\underbrace{0,\ldots,0}_{m_0+1},\underbrace{1,\ldots,1}_{m_1-1},\ldots), \quad \text{т.е.}$$

$$f[m_0,m_1,\ldots,m_{k-1}] = f[m_0+1,m_1-1,\ldots,m_{k-1}].$$

Пользуясь этим равенством, мы получаем

$$f[m_0,m_1,\ldots,m_{k-1}] = f[m_0+1,m_1-1,\ldots,m_{k-1}] = \ldots = f[m_0+m_1,0,$$

$$m_2,\ldots,m_{k-1}] = \ldots = f[m_0 + m_1 + \ldots + m_{k-1},0,\ldots,0] =$$

$$= f[n,0,\ldots,0].$$

Отсюда следует, что функция является константой.

3. ПЕРЕЧИСЛЕНИЕ СИММЕТРИЧЕСКИХ ФУНКЦИЙ ПРЕДПОЛНЫХ КЛАССОВ В P_3

Целью статьи является нахождение чисел $k(X)_s(n)$ для всех предполных классов трехзначной логики.

ТЕОРЕМА 4. $k(L)_s(n) = \begin{cases} 3 & \text{для } n = 0 \\ 9 & \text{для } n \geq 1 . \end{cases}$

Доказательство. Пусть $f(x_1,\ldots,x_n) = a_0 + a_1x_1 + \ldots + a_nx_n \pmod 3$. Из $f[1,n-1,0] = a_0 + a_1 = \ldots = a_0 + a_n$ следует, что $a_i = a_j$ $(1 \leq i, j \leq n)$. Этому классу принадлежат симметрические функции $0,1,2,\ x_1 + \ldots + x_n$, $1 + (x_1 + \ldots + x_n)$, $2+(x_1 + \ldots + x_n)$, $2(x_1 + \ldots + x_n)$, $1+2(x_1 + \ldots + x_n)$, $2+2(x_1 + \ldots + x_n)$.

ТЕОРЕМА 5.

$$k(V)_s(n) = \begin{cases} 0 & \text{для} \quad n=3m \\ 3^{\frac{1}{3}\binom{n+2}{2}} & \text{для} \quad n=3m+1 \quad \text{и} \quad n=3m+2. \end{cases}$$

Доказательство. Рассмотрим две возможности.

1. $n = 3m$. Пусть $f[m,m,m] = a$. Поскольку функция f самодвойственна, из $f[m_0,m_1,m_2] = a$ следует $f[m_1,m_2,m_0] = a+1$. Поэтому из $f[m,m,m] = a$ следует $f[m,m,m] = a + 1$. Это противоречие.

2. $n \neq 3m$. Из $f[m_0,m_1,m_2] = a$ следует $f[m_1,m_2,m_0] = a + 1$ и $f[m_2,m_0,m_1] = a + 2$. Утверждение следует из того, что наборы $[m_0,m_1,m_2]$, $[m_1,m_2,m_0]$ и $[m_2,m_0,m_1]$ попарно различные.

ОПРЕДЕЛЕНИЕ 5. Пусть $\{i,j,m\} = \{0,1,2\}$. Обозначим через $D(i,j)$ множество функций f, которых $f(x_1,\ldots,x_n) \neq m$ для всех наборов $(x_1,\ldots,x_n)$

ТЕОРЕМА 6.

$$k(T)_s(n) = \begin{cases} 27 & \text{для} \quad n = 1 \\ 3.2^{\binom{n+2}{2}} - 3 & \text{для} \quad n \neq 1. \end{cases}$$

Из определения класса T следует $T = D(0,1) \cup D(0,2) \cup D(1,2)$ $(n \geq 2)$. Число n-местных симметрических функций множества $D(i,j)$ равно $2^{\binom{n+2}{2}}$. Утверждение следует из того, что пересечение $D(i,j)$ и $D(j,m)$ содержит только константу j.

ТЕОРЕМА 7.

$$k(T_0)_s(n) = k(T_1)_s(n) = k(T_2)_s(n) = 3^{\binom{n+2}{2}-1} .$$

Доказательство. Следует из

$$f \in T_0 \iff f[n,0,0] = 0 .$$

ТЕОРЕМА 8.

$$k(T_{01})_s(n) = k(T_{02})_s(n) = k(T_{12})_s(n) = 2^{n+1}\cdot 3^{\binom{n+2}{2}-n-1}$$

Доказательство. Следует из

$$f \in T_{01} \Leftrightarrow f[m_0,m_1,0] \in \{0,1\}(m_0 + m_1 = n).$$

ТЕОРЕМА 9.

$$k(U_{01})_s(n) = k(U_{02})_s(n) = k(U_{12})_s(n) = \prod_{i=1}^{n+1} (2^i + 1).$$

Доказательство. Для фиксированного m_2 существует $n-m_2+1$ наборов $[m_0,m_1,m_2]$ $(m_0+m_1 = n-m_2)$. Из определения U_{01} следует, что значение функции для каждого из этих наборов принедлежит множеству $\{0,1\}$, или $\equiv 2$. Поэтому существует $2^{n-m_2+1}+1$ возможностей для значений $f[m_0,m_1,m_2]$ при фиксированом m_2. Утверждение теоремы следует из независимости значений функции для различных m_2.

ТЕОРЕМА 10. ([4], [5])

$$k(M_1)_s(n) = k(M_0)_s(n) = k(M_2)_s(n) = \binom{2n+3}{n+1} .$$

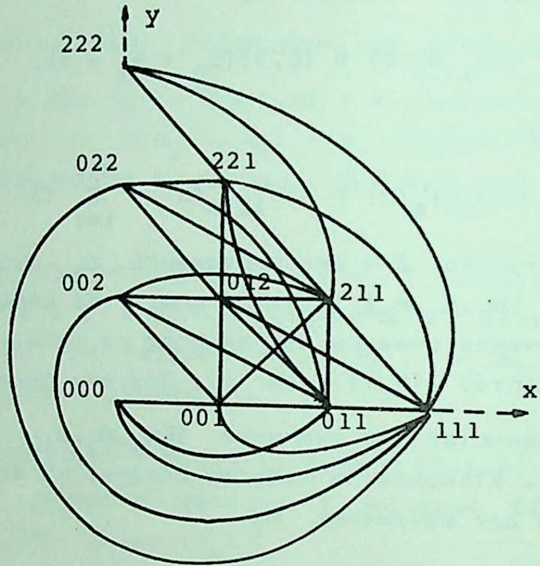
ТЕОРЕМА 11.

$$k(B_1)_s(n) = k(B_0)_s(n) = k(B_2)_s(n) =$$

$$= 1 + 2 \sum_{\substack{1 \le \sum_{s=1}^{n+1} se_s \le n+1 \\ e_s \ge 0}} \left(\frac{\left(\sum_{s=1}^{n+1} e_s\right)!}{\prod_{s=1}^{n+1} e_s!} \prod_{s=1}^{n+1} \left(2^{\binom{s+1}{2}} - 2^{\binom{s}{2}}\right)^{e_s} \right)$$

Доказательство. Определим $k(B_1)_s(n)$. Изобразим неориентированый граф наборов $[z,x,y]$ в системе координат. Соединим только те наборы $\alpha' = (\alpha_1',\ldots,\alpha_n')$ и $\alpha'' = (\alpha_1'',\ldots,\alpha_n'')$, для которых существует набор $(\alpha_1,\ldots,\alpha_n)$,

$\alpha_i \in \{0,2\}$, такой что $\alpha_i' \neq \alpha_i, \alpha_i'' \neq \alpha_i$ $(1 \leq i \leq n)$. Для соединенных наборов α' и α'', $f(\alpha') \neq 0$ или $f(\alpha'') \neq 2$. На рисунке представлен граф для $n = 3$.



Из определения B_1 следует, что наборы $[z', x', y']$ и $[z'', x'', y'']$ соединенны тогда и только тогда, когда $y'' \leq x' + y'$ и $y' \leq x'' + y''$.

Граф $(n + 1)$ - членных наборов получается из n-членных наборов следующим образом:

1) n-членному набору $[z,x,y]$ отвечает $n+1$-членный набор $[z+1,x,y]$.
2) Граф дополняется вершинамы $[0,x,y]$, $x+y = n+1$.
3) Вершина $[0,x,y]$ $(x+y = n+1)$ соединяется с вершинами $[z', x', y']$ $(z' + x' + y' = n + 1)$ для которых $z' \leq x$, т.е $n+1 \leq x + x' + y'$.

Пусть k_0 наиманьшее число нулей, для которого существует набор $[z,x,y]$ $(z+x+y = n)$ такой, что $f[z,x,y] = 0$. т.е. $k_0 = \min\limits_z (f[z,x,y] = 0)$. Эсли $f[z,x,y] \neq 0$ для всех наборов пусть $k_0 = n + 1$.

Подобным образом пусть $k_2 = \min_z (f[z,x,y] = 2)$.

Обозначим через $g_{k_0,k_2}(n)$ число симметрических n-местных функций f для которых $\min_z (f[z,x,y] = 0) = k_0$ и $\min_z (f[z,x,y] = 2) = k_2$. Каждая функция f из $g_{k_0,k_2}(n)$ переходом с графа n-членных наборов к графе n+1-членных наборов определяет:

1) одну функцию f' из $g_{k_0+1,k_2+1}(n+1)$ если $f'[0,x,y] = 1$ для всех $x,y,x+y = n+1$.

2) функции f' из $g_{k_0+1,0}(n+1)$, для которых

$$f'[0,x,y] = \begin{cases} 1 & \text{для} \quad y \leq n-k_0 \\ \in \{1,2\} & \text{для} \quad y > n-k_0 \end{cases}$$

и для которых существует набор $[0,x,y]$ такой, что $f'[0,x,y] = 2$. Число таких функций f' равно $2^{k_0+1} - 1$.

3) функции f' из $g_{0,k_2+1}(n+1)$, для которых

$$f'[0,x,y] = \begin{cases} 1 & \text{для} \quad y \leq n-k_2 \\ \in \{1,0\} & \text{для} \quad y > n-k_2 \end{cases}$$

и для которых существует набор $[0,x,y]$ такой, что $f'[0,x,y] = 0$.

Число таких функций f' равно $2^{k_2+1}-1$.

Из этого замечания вытекают следующие равенства.

(1) $$g_{k_0+1,k_2+1}(n+1) = g_{k_0,k_2}(n)\ , \quad k_0,\ k_2 \geq 0\ ,$$

$$g_{k_0+1,0}(n+1) = (2^{k_0+1} - 1) \sum_{k_2=0}^{n+1} g_{k_0,k_2}(n)\ ,$$

$$g_{k_2,k_0}(n) = g_{k_0,k_2}(n)$$

$$g_{0,0}(n) = 0,\ g_{0,0}(0) = 0,\ g_{1,1}(0) = 1,\ g_{1,0}(0) = 1.$$

Из этой системы для $k_0 > k_2$ получаем

(2) $$g_{k_0,k_2}(n) = g_{k_0-1,k_2-1}(n) = \dots = g_{k_0-k_2,0}(n-k_2)$$

Из определения $g_{k_0,k_2}(n)$ следует

(3) $$k(B_1)_s(n) = \sum_{0\le k_0,k_2\le n+1} g_{k_0,k_2}(n) \quad (n \ge 0) .$$

В таблице представлены числа $k(B_1)_s(n)$ для $n \le 4$.

n	0	1	2	3	4
$k(B_1)_s(n)$	3	17	155	2409	72721

Пусть $h(k,n) = g_{k,0}(n)$.
Из (1), (2) и (3) следует

(4) $$k(B_1)_s(n+1) = \sum_{0\le k_0,k_2\le n+2} g_{k_0,k_2}(n+1) =$$

$$= g_{0,0}(n+1)+2\sum_{1\le k_0\le n+2} g_{k_0,0}(n+1) + \sum_{1\le k_0,k_2\le n+1} g_{k_0,k_2}(n+1)=$$

$$= 2\sum_{1\le k\le n+2} h(k,n+1) + \sum_{0\le k_0,k_2\le n+1} g_{k_0,k_2}(n) =$$

$$=2\sum_{1\le k\le n+2} h(k,n+1) + k(B_1)_s(n).$$

Из (1) и (2) получаем

$$h(k,n) = g_{k,0}(n) = (2^k - 1)\sum_{k_2=0}^{n} g_{k-1,k_2}(n-1) =$$

$$= (2^k-1)(g_{k-1,0}(n-1) + \sum_{k_2=1}^{n} g_{k-1,k_2}(n-1)) =$$

$$= (2^k - 1)(h(k-1,n-1) + \sum_{k_2=1}^{n} g_{k-2,k_2-1}(n-2)) =$$

$$= (2^k - 1)(h(k-1,n-1) + \frac{h(k-1,n-1)}{2^{k-1}-1}) =$$

$$=2^{k-1}\frac{2^k-1}{2^{k-1}-1}\,h(k-1,n-1) = \frac{2^k-1}{2^{k-1}-1}\,2^{k-1}\cdot\frac{2^{k-1}-1}{2^{k-2}-1}\,2^{k-2}h(k-2,n-2) =$$

$$= \ldots = (2^k-1)2^{k-1}\cdot 2^{k-2}\ldots\frac{2^1}{2^1-1}\cdot h(1,n-k+1).$$

Отсюда получаем

$$(5)\qquad h(k,n) = \left(2^{\binom{k+1}{2}} - 2^{\binom{k}{2}}\right) h(1,n-k+1).$$

Из $h(1,n) = g_{1,0}(n) = (2^1-1)\sum\limits_{k_2=0}^{n} g_{0,k_2}(n-1) = \sum\limits_{k=1}^{n} h(k,n-1)$

следует, пользуясь (4),

$$(6)\qquad k(B_1)_s(n-1) = k(B_1)_s(n-2) + 2h(1,n)\ ,\qquad (n \geq 2)$$

$$(7)\quad h(1,n) = \sum_{k=1}^{n} h(k,n-1) = \sum_{k=1}^{n}\left(2^{\binom{k+1}{2}} - 2^{\binom{k}{2}}\right)h(1,n-k)\ \ (\text{из } (5))\ .$$

Из (6) следует

$$\sum_{i=0}^{n} k(B_1)_s(i) = 2\sum_{i=0}^{n} h(1,i+1) + \sum_{i=0}^{n} k(B_1)_s(i-1),\ \text{т.е.}$$

$$(8)\qquad k(B_1)_s(n) = 1+2\sum_{k=1}^{n} h(1,k)\ .$$

Пусть $a_k = 2^{\binom{k+1}{2}} - 2^{\binom{k}{2}}$.

Для определения $h(1,n)$ пользуемся равенством (7).

$$h(1,n) = \sum_{k=1}^{n} a_k h(1,n-k)$$

$$h(1,1) = a_1$$

$$h(1,2) = a_1\cdot a_1 + a_2 = a_1^2 + a_2$$

$$h(1,3) = (a_1^2 + a_2)\cdot a_1 + a_1a_2 + a_3 = a_1^3 + 2a_1a_2 + a_3$$

$$h(1,4) = (a_1^3 + 2a_1a_2 + a_3)\cdot a_1 + (a_1^2 + a_2)\cdot a_2 + a_1a_3 + a_4 =$$

$$= a_1^4 + 3a_1^2a_2 + 2a_1a_3 + a_2^2 + a_4\ .$$

Индукцией легко можно доказать, что $h(1,n)$ содержит все возможные произведения $a_{i_1} a_{i_2} \dots a_{i_j}$, для которых $i_1 + i_2 + \dots + i_j = n$, $i_1, i_2, \dots, i_j > 0$.
Индекси $i_1, i_2, \dots, i_j$ отвечают разбиениям числа n, т.е. представлениям числа n в виде суммы целых положительных чисел с существенным порядком.
Обозначим через e_s число i_s в разбиении числа n.
Каждом разбиении $i_1 + i_2 + \dots + i_j = n$ для которого $i_1 \leq i_2 \leq \dots \leq i_s$ отвечает $\frac{(e_i + \dots + e_j)!}{e_1! e_2! \dots e_j!}$ разбиений с учетом порядка слагаемых. Из этих замечаний следует

$$(9) \quad h(1,n) = \sum_{\substack{e_1+2e_2+\dots+ne_n=0 \\ e_j \geq 0}} \frac{(e_1+\dots+e_n)!}{e_1! \dots e_n!} a_1^{e_1} \dots a_n^{e_n} .$$

Утверждение теоремы 11 следует из (8) и (9).
Число симметрических невырожденных функций каждого предполного класса можно легко определить пользуясь теоремой 3 и теоремамы 4 - 11.

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REZIME

PREBROJAVANJE SIMETRIČNIH FUNKCIJA SKORO KOMPLETNIH SKUPOVA TROZNAČNE LOGIKE

U radu je odredjen broj n-arnih simetričnih funkcija za svaki od 18 skoro kompletnih skupova troznačne logike.

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AN EXTENSION OF THE LISPKIT LISP-LANGUAGE VERSION ARL BY THE GENERALIZED FUNCTIONS OF SOME PRIMITIVE FUNCTIONS AND THEIR IMPLEMENTATION

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ABSTRACT

The paper presents the implementation of an extension of the LISPKIT LISP-language version ARL by the following primitive generalized functions of the standard LISP-language:

(PLUS i1 ... ik)   k>=1
(TIMES i1 ... ik)
(OR ℓ1 ... ℓk)
(AND ℓ1 ... ℓk)
(COND (ℓ1 e1) ... (ℓk ek)

i1,...,ik are integer well-formed expressions,
ℓ1,...,ℓk are logical well-formed expressions,
e1,...,ek are arbitrary well-formed expressions,

* *The same paper, but under the title:*
"An extension of LISPKIT LISP-language and its implementation" was reported at the Seventh Congress of Balkan Mathematicians in Athens, 1983.

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and their degenerated cases:

(PLUS)
(TIMES)
(OR)
(AND)
(COND)

in an indirect and direct way.

The implementation was performed in the LISPKIT LISP-language version ARL.

INTRODUCTION

P. Henderson [1] has described LISPKIT LISP-language and the principles of design of the LISPKIT LISP-system.

The Group for Functional Programming of the Institute of Mathematics, Faculty of Science University of Novi Sad [2], [3], [4], [5], [6] using the results of P. Henderson, and at the same time giving a number of its own solutions has constructed a LISPKIT LISP-system in FORTRAN-language and installed it in a DELTA 11/340 Computer System.

During work on solving various problems of symbolic data processing, for example:

- Symbolic differation of functions,
- Simulating of finite-state and pushdown automata,
- Match-functions etc.,

the need for the introduction of:

A) new primitive:
- Arithmetic functions: EXP
- Relational functions: LT, NE, GE, GT,

B) The generalized functions of the following primitive functions: ADD, MUL, DIS, CON, IF.

is appeared.

THE GENERALIZED FUNCTIONS

Let us generalize the following primitive functions of the LISPKIT LISP-language version ARL:

(ADD i1 i2)
(MUL i1 i2)
(DIS ℓ1 ℓ2)
(CON ℓ1 ℓ2
(IF ℓ e1 e2)

The generalization consists of taking an arbitrary number of arguments of function, k>=0.

For k = 0:
(ADD)
(MUL)
(DIS)
(CON)
(IIFF)

For k>= 1:

(ADD i1 ... ik)
(MUL i1 ... ik)
(DIS ℓ1 ... ℓk)
(CON ℓ1 ... ℓk)
(IIFF (ℓ1 e1) ... (ℓ1 ek))

For k = 0 the presented generalized functions are the following constants:

- 0
- 1
- F
- T
- Ω

Ω is an arbitable, usually NIL value. Ω serves for reserving the memory space for a pointer.

In the case of IF-function the meaning of the argument

was changed. Because of that, the new name of the function: IIFF was introduced.

The given generalized functions are not the functions in the usual mathematical sense. In mathematics it is usual that a function has a fixed number of arguments.

For k>= 1, the given generalized functions are sequentially identical with the following primitive generalized functions of the standard LISP-language:

```
(PLUS i1 ... ik)
(TIMES i1 ... ik)
(OR ℓ1 ... ℓk)
(AND ℓ1 ... ℓk)
(COND (ℓ1 e1) ... (ℓk ek))
```

The presented primitive generalized functions of the standard LISP-language can be defined in the following non-formal way by means of corresponding primitive functions of the LISPKIT LISP-language version ARL:

```
(PLUS i1 ... ik) = (ADD i1 ( ... (ADD ik (QUOTE 0)) ... ))
(TIMES i1 ... ik) = (MUL i1 ( ... (MUL ik (QUOTE 1)) ... ))
(OR ℓ1 ... ℓk) = (DIS ℓ1 ( ... (DIS ℓk (QUOTE F)) ... ))
(AND ℓ1 ... ℓk) = (CON ℓ1 ( ... (CON ℓk (QUOTE T)) ... ))
(COND (ℓ1 e1) ... (ℓk ek)) = (IF ℓ1 e1 ( ... (IF ℓk ek ) ... ))
```

THE IMPLEMENTATION OF GENERALIZED FUNCTIONS

The generalized functions cannot be implemented as the user functions. The definitional expression:

```
(LAMBDA(x1 ... xk)e)
```

of the LISPKIT LISP-language requires, known beforehand, the fixed number of arguments of functions which is to be defined.

The generalized functions can be implemented exclusively as the primitive functions, that is, they must be built-in into the system. Each building-in of functions into the system

requires the modification of the system.

The implementation of the generalized functions requires only the modification of the translator, that is, the part of the LISPKIT LISP-system which executes the translation of the program from the LISPKIT LISP-language in the machine language of the SECD-machine. The modification of the program simulator of the SECD-machine is not necessary.

There are two methods of implementation of the generalized functions:

- indirect and
- direct.

In the case of indirect implementation the generalized function is translated in the composition of the corresponding primitive function and this composition is further translated into machine language of the SECD-machine by means of the same translator.

In the case of direct implementation, the generalized function is translated straight into the machine language of the SECD-machine.

The indirect implementation does not require a knowledge of:

- the machine language of the SECD-machine and
- the code of translation.

Indirect implementation is:

- more simple and
- slower than

direct implementation.

The indirect translation of the generalized functions can also be realized in the form of the special pre-translator.

THE IMPLEMENTATION OF PLUS-FUNCTION

The part of the translator which serves for translation of the ADD-function is:

```
(1)...  (IF (EQ (CAR E)
              (QUOTE ADD)
          )
          (COMP (CAR (CDR E))
                N
                (COMP (CAR (CDR (CDR E)))
                      N
                      (CONS (QUOTE ADD)
                            C
                      )
                )
          )  ...
```

A) Indirect implementation

The degenerated PLUS-function is translated in:

(PLUS) ⟶ (QUOTE 0)

The non-degenerated PLUS-function is translated in the composition of ADD-functions:

(PLUS i1 ... ik) → (ADD i1(... (ADD ik(QUOTE 0))...))

The obtained translations are further translated in the machine language of the SECD-machine by means of the same translator.

The translator is extended with:

```
(2)...  (IF (EQ (CAR E)
              (QUOTE PLUS)
          )
          (COMP (PLUSFUN (CDR E))
                N
                C
          )...
```

where is:

```
(3)...  (PLUSFUN LAMBDA (E)
          (IF (EQ E
                  (QUOTE NIL)
              )
              (QUOTE (QUOTE 0))
              (SPOJ3 (QUOTE ADD)
                     (CAR E)
                     (PLUSFUN (CDR E))
              )
          )
        )...
(4)...  (SPOJ3 LAMBDA (A B C)
          (CONS A
                (CONS B
                      (CONS C
                            (QUOTE NIL)
                      )
                )
          )
        )...
```

It is necessary that the translator still contains (1).

B) Direct implementation

The degenerated PLUS-functions is translated in:

(PLUS) → (LDC 0.c)

The non-degenerated PLUS-function is translated in:

(PLUS i1 ... ik) → (LD i1 ... LD ik LOC O ADD ... ADD.c)

The translator is extended with:

```
(2')... (IF (EQ (CAR E)
             (QUOTE PLUS)
             )
             (PLUSFUN (CDR E)
                      N
                      C
             )...
```

where is:

```
(3')...  (PLUSFUN LAMBDA (E N C)
           (IF (EQ E
                   (QUOTE NIL)
               )
               (CONS (QUOTE LOC)
                     (CONS (QUOTE 0)
                           C
                     )
               )
               (COMP (CAR E)
                     N
                     (PLUSFUN (CDR E)
                              N
                              (CONS (QUOTE ADD)
                                    C))) ))...
```

The translator can, but need not, contain (1), depending on whether one wishes or not to keep ADD-function.

The implementation of TIMES, OR and AND function.

The given functions are completely implemented in a similar way as the PLUS-function.

The analogous expressions of (1), (2), (3), (2'), (3') are the following:

```
(1")... (IF (EQ (CAR E)
                (QUOTE MUL)
            )
            (COMP (CAR (CDR E))
                  N
                  (COMP (CAR (CDR (CDR E)))
                        N
                        (CONS (QUOTE MUL)
                              C
                        )
                  )
            )
```

```
(IF (EQ (CAR E)
        (QUOTE DIS)
    )
    (COMP (CAR (CDR E))
          N
          (COMP (CAR (CDR (CDR E)))
                N
                (CONS (QUOTE DIS)
                      C
                )
          )
    )
(IF (EQ (CAR E)
        (QUOTE CON)
    )
    (COMP (CAR (CDR E))
          N
          (COMP (CAR (CDR (CDR E)))
                N
                (CONS (QUOTE CON)
                      C
                )
          )
    )...
```

(2")...
```
(IF (EQ (CAR E)(QUOTE TIMES))
    (COMP (TIMESFUN (CDR E)) N C)
(IF (EQ (CAR E)(QUOTE OR))
    (COMP (ORFUN (CDR E)) N C)
(IF (EQ (CAR E)(QUOTE AND))
    (COMP (ANDFUN (CDR E) N C)...
```

(3")...
```
(TIMESFUN LAMBDA (E)
   (IF (EQ E (QUOTE NIL))
       (QUOTE (QUOTE 1))
       (SPOJ3 (QUOTE MUL)(CAR E)(TIMESFUN (CDR E))) ))
  (ORFUN LAMBDA (E)
    (IF (EQ E (QUOTE NIL))
```

```
            (QUOTE (QUOTE F))
            (SPOJ3 (QUOTE OR)(CAR E)(ORFUN (CDR E))) ))
      (ANDFUN LAMBDA (E)
        (IF (EQ E (QUOTE NIL))
            (QUOTE (QUOTE T))
            (SPOJ3 (QUOTE AND)(CAR E)(ANDFUN (CDR E))) ))...
```

```
(2''')...  (IF (EQ (CAR E)(QUOTE TIMES))
               (TIMESFUN (CDR E) N C)
           (IF (EQ (CAR E)(QUOTE OR))
               (ORFUN (CDR E) N C)
           (IF (EQ (CAR E)(QUOTE AND))
               (ANDFUN (CDR E) N C)...
```

```
(3''')...  (TIMESFUN LAMBDA (E N C)
             (IF (EQ E (QUOTE NIL))
                 (CONS (QUOTE LDC)(CONS (QUOTE 1) C ))
                 (COMP (CAR E) N (TIMESFUN (CDR E)
                                           N
                                           (CONS (QUOTE MUL)
                                                 C
                                           )
                                 )
                 )
             )
           )
           (ORFUN LAMBDA (E N C)
             (IF (EQ E (QUOTE NIL))
                 (CONS (QUOTE LDC)(CONS (QUOTE F) C ))
                 (COMP (CAR E) N (ORFUN (CDR E)
                                        N
                                        (CONS (QUOTE OR)
                                              C
                                        )
                                 )
                 )
             )
           )
```

```
(ANDFUN LAMBDA (E N C)
  (IF (EQ E (QUOTE NIL))
      (CONS (QUOTE LDC)(CONS (QUOTE T) C ))
      (COMP (CAR E) N (ANDFUN (CDR E)
                              N
                              (CONS (QUOTE AND)
                                    C
                              )
                      )
      )
  )
)...
```

Implementation of COND-function

The part of the translator which serves for the translation of the IF-function is:

```
(1)...  (IF (EQ (CAR E)
            (QUOTE IF)
            )
            (COMP (CAR (CDR E))
                  N
                  (CONS (QUOTE SEL)
                        (CONS (COMP (CAR (CDR (CDR E)))
                                    N
                                    (QUOTE (JOIN))
                                    )
                                    (CONS (COMP (CAR (CDR (CDR (CDR E))))
                                                N
                                                (QUOTE (JOIN))
                                          )
                                          C
                                    )
                        )
                  )
        )...
```

A) Indirect implementation

The degenerated COND-function is translated in:

(COND) → (QUOTE Ω)

The non-degenerated COND-function is translated in the composition of the IF-function:

(COND (ℓ1 e1) ...(ℓk ek)) → (IF ℓ1 e1(...(IF ℓk ek Ω) ...))

The obtained translations are further translated in the machine language of the SECD-machine by means the same translator. The translator is extended with:

```
(2)...  (IF (EQ (CAR E)
                (QUOTE COND)
            )
            (COMP (CONDFUN (CDR E))
                  N
                  C
            )...
```

where is:

```
(3)...  (CONDFUN LAMBDA (E)
          (IF (EQ E
                  (QUOTE NIL)
              )
              (QUOTE (QUOTE Ω))
              (SPOJ4 (QUOTE IF))
                     (CAR (CAR E))
                     (CAR (CAR (CDR E)))
                     (CONDFUN (CDR E))
              )
          )
        )...
(4)...  (SPOJ4 LAMBDA (A B C D)
          (CONS A
                (CONS B
                      (CONS C
                            (CONS D
```

```
                                        (QUOTE NIL)
                                  )
                           )
                    )
             )...
```

It is necessary that the translator still contains (1).

B) Direct implementation

The generated COND-function is translated in:

(COND) → (LDC Ω.c)

The non-degenerated COND-function is translated in:

(COND (ℓ1 e1) ...(ℓk ek)) →

(LD ℓ1 SEL (LD e1 JOIN)(...(LD ℓk SEL(LD ek JOIN)(LDC Ω JOIN)JOIN)...))

The translator is extended with:

```
(2')    (IF (EQ (CAR E)
                (QUOTE COND)
            )
            (CONDFUN (CDR E)
                     N
                     C
            )...
```

where is:

```
(3')... (CONDFUN LAMBDA (E N C)
           (IF (EQ E
                   (QUOTE NIL)
               (CONS (QUOTE LDC)
                     (CONS (QUOTE Ω)
                           C
                     )
               )
           (IF (EQ (CDR E)
                   (QUOTE NIL)
               )
```

```
        (COMP (CAR (CDR (CAR E)))
              N
              C
        )
        (COMP (CAR (CAR E))
              N
              (CONS (QUOTE SEL)
                    (CONS (COMP (CAR (CDR (CAR E)))
                                N
                                (QUOTE (JOIN))
                          )
                          (CONS (CONDFUN (CDR E)
                                         N
                                         (QUOTE (JOIN))
                                )
                                C
                          )
                    )
              )
        )
    )...
```

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REZIME

O PROŠIRENJU LISPKIT LISP-JEZIKA VERZIJE ARL POMOĆU UOPŠTENIH FUNKCIJA NEKIH PRIMITIVNIH FUNKCIJA I NJIHOVA IMPLEMENTACIJA

U radu je izvršeno uopštavanje sledećih primitivnih funkcija LISPKIT LISP-jezika verzije ARL:

(ADD i1 i2)
(MUL i1 i2)
(DIS ℓ1 ℓ2)
(CON ℓ1 ℓ2)
(IF ℓ e1 e2)

-i1,i2 su celobrojni valjani izrazi,
-ℓ,ℓ1,ℓ2 su logički valjani izrazi i
-e1,e2 su proizvoljni valjani izrazi.

Uopštavanje se sastoji u tome da se dozvoli proizvoljan broj k,k>=0, argumenta funkcije:

za k=0:	za k>=1
(ADD)	(ADD i1 ... ik)
(MUL)	(MUL i1 ... ik)
(DIS)	(DIS ℓ1 ... ℓk)
(CON)	(CON ℓ1 ... ℓk)
(IIFF)	(IIFF (ℓ1 e1) ... (ℓk...ek))

-il,...,ik su celobrojni valjani izrazi
-ℓ1,...,ℓk su logički valjani izrazi i
-e1,...,ek su proizvoljni valjani izrazi.

Za k>=1, navedene uopštene funkcije su redom identične sa sledećim primitivnim uopštenim funkcijama standardnog LISP-jezika:

(PLUS il ... ik)
(TIMES il ... ik)
(OR ℓ1 ... ℓk)
(AND ℓ1 ... ℓk)
(COND (ℓ1 e1) ... (ℓk ek)).

Navedene primitivne uopštene funkcije standardnog LISP-jezika, zajedno sa njihovim degenerisanim slučajem (PLUS), (TIMES), (OR), (AND) i (COND), implementirane su na indirektan i direktan način na LISPKIT LISP-jeziku verzija ARL.

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ON n-FINITE FORCING

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ABSTRACT

The main result of this paper is that n-finite forcing companion (of a given theory) can be obtained by an application of (Robinson's) finite forcing (Corollary 3.3). Hence, in particular, it follows: for any theory T defined in a language L there exists its extension defined in a suitable expanded language L', the finite and n-finite forcing companions of which coincide (Theorem 3.8).

INTRODUCTION

In [2] we concluded that the main properties of Robinson's finite forcing are naturally transmitted (in the sense of their "translation for n") to n-finite forcing . (We did not, completely justified make an efort to give the complete proofs for all of them, down to the last one, were inspired by the corresponding ones for finite forcing). Since, however, on that occasion we used, without additional explanation, also the result from [4] (1,5 in [2]) (of a more general character), which is not directly applicable with regard to the fact that for condition we took subsets of the set (C_n) of all senten-

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ces which are equivalent to sentences in prenex normal form with at most n blocks of quantifiers (it is not fulfilled: $\phi \in C_n \rightarrow \text{sub}(\phi) \subset C_n$) this time we shall deal with this more thoroughly. Simultaneously we shall point out the part of the "assortment", elements of which can be taken equally for a set of the condition, as well as the possible simplifications and improvements of some proofs and propositions. We shall also correct two results from [2].

n-finite forcing itself, for $n > 0$, being considered from a purely "technical" aspect does not offer much new, namely, n-finite forcing companion can be obtained by an application of Robinson's finite forcing as well. We are of the opinion howeover, that doing research on its properties and particularly on properties of the corresponding forcing companion is useful because of many reasons (and from almost all other points of view).

§ 0. Throughout this paper T is a fixed (but otherwise arbitrary) theory defined in a finite language of first order L. L(A) is a normal extension of the language L (i.e. $L(A) = L \cup A$ where A is an infinite set (according to the need of sufficiently large cardinality) of new constants and $L \cap A = \emptyset$). Of course as long as we are in the sphere of syntax we can assume that A is countable infinite.

For the basic logical symbols we take $\neg$ (negation), $\wedge$ (conjuction) and $\exists$ (existential quantifier) (the others are defined by the basic ones in the standard way). FORM (L), SENT(L), AT(L) are, successively, the sets of all formulas, sentences and atomic sentences of the language L. Naturally if F is a set of formulas, SENT(F) will be the set $\{\phi \in F | \phi$ is a sentence$\}$. Φ_n ($\Phi_n(A)$), $n \geq 0$ is the set $\{\phi \in \text{FORM}(L)\ (\text{FORM}(L(A)))|\ \phi$ is logically equivalent to a formula of the language L(L(A)) in prenex normal form with at most n blocks of quantifiers$\}$, and $C_n = \{p | p$ is a finite subset of the set $\text{SENT}(\Phi_n(A))$ which is consistent with $T\}$. $\mu(T)$ is the class of all models of the theory T and $T \cap \Pi_{n+1}$ (Π_{n+1} segment of T) is $\{\phi \in \text{SENT}(\Phi_n) | T \vdash \phi\}$.

We shall assume a knowledge of the definitions of "forcing notions" (forcing system, forcing relation and so on) as well as of their basic general properties (see [1], [5]). Yet, for the coherence of the text we shall quote (for us this time, most relevant) results (more about them as well as about their more complete version can be found in [3], [5]) remarking that in both cases we assume the standard set of logical axioms (for instance the one given in [5]),

THEOREM 0.1. *Let* $\langle C, \Vdash, L\rangle$ *be a forcing system where* L *is a finite language of arbitrary cardinality (i.e. the set of nonlogical symbols is of arbitrary cardinality). Then for each* $p \in C$ *it holds that:*

(1) $T^C[p] = \{\phi \in SENT(L) \mid p \Vdash \neg\neg\phi\}$ *is a consistent and deductively closed set* ($T^C[p] \vdash \phi$ *implies* $\phi \in T^C[p]$);

(2) *If* $\phi(v_o,\ldots,v_m)$ *is a logically valid formula (that is,* $\vdash_L \phi(v_o,\ldots,v_m)$) *then for any closed terms* $t_o,\ldots,t_m$ $\phi(t_o,\ldots,t_m) \in T^C[p]$.

THEOREM 0.2. *Let* T *be a theory of a finite language* L, A *an infinite set of new constants and let* $F \subseteq FORM(L(A))$ *satisfy the following conditions:*

(1) $\phi \in F$ *implies* $sub(\phi) \subseteq F$;

(2) $\phi \in F$ *implies* $\neg\phi \in F$

and (3) *If* $\phi(v) \in F$ *and* t *is a closed term of the language* L(A) *then, also,* $\phi(t) \in F$.

If C is the set $\{p \subset SENT(F) \mid |p| < \aleph_o$ and $T \cup p$ is consistent$\}$ partially ordered by inclusion and the relation $\Vdash$ ($\subseteq C \times SENT(L(A))$) is determined for $\phi \in AT(L(A))$ by

$$p \Vdash \phi \text{ if and only if } \phi \in p$$

and otherwise is defined with regard to the basic logic symbols as a forcing relation, then for each $p \in C$ and each $\phi \in SENT(F)$ it holds that:

I (a) From $\phi \in p$ follows $p \Vdash \neg\neg\phi$; (b) $p \Vdash \neg\neg\phi$ implies $T \cup p \nvdash \neg\phi$; (c) if $T \cup p \nvdash \neg\phi$ then there exists an element $q \in C$ such that $p \cup \{\phi\} \subseteq q$.

(REMARK: $\Vdash$ can, but does not have to be a forcing relation; of course, a sufficient condition for $\Vdash$ to be a forcing relation is that F contains all atomic formulas);

II $p \Vdash \neg\neg\phi$ if and only if $T \cup p \vdash \phi$.

n-finite forcing is a triple $\langle C_n, \Vdash_n, L(A)\rangle$ where the forcing relation is determined (with regard to the basic logical symbols) by:

$$p \Vdash_n \phi \text{ if and only if } \phi \in p \text{ for } \phi \in AT(L(A))$$

Let M be a model of the language L and f a mapping of A into M. We shall say that $\langle A,f\rangle$ is an assignment of constants to M if $f(A)$ is a generative set for M (i.e. no proper substructure of M contains $f(A)$) ([1]). Therefore if $\langle A,f\rangle$ is an assignment of constants to M and each $a \in A$ is interpreted in M by $f(a)$ then each element of M is, let us say, denoted by (at least) one closed term (whose interpretation it is) of the language $L(A)$. Hence $M \in \mu(T \cap \Pi_{n+1})$ if and only if each finite subset of n-elementary diagram of the model M $D_n(M) = \{\phi \in SENT(\Phi_n(A)) \mid M \models \phi\}$ is a condition. By $F_{\langle A,f\rangle}(M)$ we denote the set $\{\phi(v_{i_0},\dots,v_{i_k}) \in FORM(L) \mid$ for any closed terms $t_0,\dots,t_k$ of the language $L(A)$ $M \models \phi(t_0,\dots,t_k)$ iff for some $p \subset D_n(M)$ $p \Vdash \phi(t_0,\dots,t_k)$. The well-known theorem says that for any two assignments of constants to M $\langle A,f\rangle, \langle A_1,f_1\rangle$ $F_{\langle A,f\rangle}(M) = F_{\langle A_1,f_1\rangle}(M)$, Thus we can write only $F(M)$. M is, let us recall that too, (T-) n finitely generic model if it is satisfied:

$$(1) \quad M \in (T \cap \Pi_{n+1})$$

and

$$(2) \quad F(M) = FORM(L)$$

§ 1. For n-finite forcing, let us say this immediately (the proof will be given in the next paragraph), assertions analogous to I and II from 0.2 hold:

THEOREM 1.1. *Let* $p \in C_n$ *and* $\phi \in \mathrm{SENT}(\Phi_n(A))$. *Then* (1) $\phi \in p$; (2) $p \Vdash_n \neg\neg\phi$; (3) $T \cup p \nvdash \neg\phi$ *and* (4) *there exists a condition* q *such that* $p \cup \{\phi\} \subseteq q$ *satisfy:*

(a) (1) $\rightarrow$ (2); (b) (2) $\rightarrow$ (3) *and* (c) (3) $\rightarrow$ (4)

COROLLARY 1.2. *For* $p \in C_n$ *and* $\phi \in \mathrm{SENT}(\Phi_n(A))$ *holds:* $p \Vdash_n \neg\neg\phi$ *if and only if* $T \cup p \vdash \phi$.

In [2] we have used more, in fact, (in truth not quite complete) Theorem 1.1 than result 1.5 there mentioned (taken over from [4]), which represents its weakened or strengthened - depending on how one looks at it-version.

The rest of this paragraph is devoted mainly to the correction and improvement of two results from [2].

THEOREM 1.3. *Let* M *be a model of the theory* $T \cap \Pi_{n+1}$ *and* $\langle A,f\rangle$ *an assignment of constants to* M. *Then the set* $F_{\langle A,f\rangle}(M)$ *contains all basic formulas and is closed under conjuction and existential quantifier.*

In particular for any Σ_n, Π_n *formula* $\phi(v_o,\ldots,v_m)$ *and any closed terms* $t_o,\ldots,t_m$ *of the language* $L(A)$

$$M \models \phi(t_o,\ldots,t_m) \text{ if and only if } p \Vdash_n \neg\neg\phi(t_o,\ldots,t_m)$$

for some $p \subset D_n(M)$.

P r o o f. The first part of the theorem obviously holds. In the proof of the second part we apply 1.1 and 1.2.

If $M \models \phi(t_o,\ldots,t_m)$ then by 1.1 $\{\phi(t_o,\ldots,t_m)\} \Vdash_n \neg\neg\phi(t_o,\ldots,t_m)$.

If for some $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi$ we have with respect to 1.2 $T \cup p \vdash \phi$. Let N be a model of T such that $M \prec_n N$ Then $N \models \phi$ and hence also $M \models \phi$.

In general $F_{<A,f>}(\mathcal{M})$ is not closed under negation and disjunction. The difference between the definition of a forcing relation given in [1], [6] and ours (which does not take disjunction as the basic logic symbol), which is otherwise eliminated by weak forcing, makes (in [1]) $F_{<A,f>}(\mathcal{M})$ closed under disjunction as well.

Keeping in mind the already mentioned theorem that $F_{<A,f>}(\mathcal{M})$ does not depend on $<A,f>$ (the proof of which is based on 1.3) we shall assume in all the following examples that f is a one-to-one and onto mapping, and use the same notation for element of M and the constant from A corresponding to it. Also we shall write $F(\mathcal{M})$ more simply.

EXAMPLE 1.4. Let T be a theory which "says" that R is a relation of the (strict) order (i.e. irreflexive, antisymmetric and transitive relation), $<M,R^{\mathcal{M}}>$ its model where $M = \{a_i \mid i \in w\}$ and $R^{\mathcal{M}} = \{<a_o,a>\}$ and let $\Vdash_o$ be (0-) finite forcing relation. Then $\exists vR(v,a_2), \exists vR(a_2,v) \in F(\mathcal{M})$ but $\mathcal{M} \nvDash \exists vR(v,a_2) \vee \exists vR(v,a_2)$ while $p \Vdash \exists vR(v,a_2) \vee \exists vR(a_2,v)$ (that is $p \Vdash \neg(\neg\exists vR(v,a_2) \wedge \neg\exists vR(a_2,v))$ for any $p \subset D_o(\mathcal{M})$. By the way, let us note that $\neg\exists vR(v,a_2) \notin F(\mathcal{M})$ for no condition $p \subset D_o(\mathcal{M})$ forces $\neg\exists vR(v,a_2)$ (as no condition $p \subset D_o(\mathcal{M})$ forces $\exists vR(v,a_2)$).

In case that $\vee$ is taken for basic logic symbol (and $\wedge$ is defined by $\vee$ and $\neg$), $F(\mathcal{M})$ is not necessarily closed under conjunction. For a demonstration of that, one can use the above example. By this we want, at the same time, to point out the error made in [2], where in paragraph 1 we accepted the definition of a forcing relation given in [4], but in the next paragraphs we used the one from [1], [6]. This demands that from 3.3 in [2] we delete "finite conjunction" along with, otherwise, wrong "countable" which should be replaced by "finite" (disjunction).

EXAMPLE 1.5(a). Let T be, as in the previous example, the theory of (strict) order and $<M,R^{\mathcal{M}}>$ its model, where

$M = \{a_i | i \in Z\} \cup \{b_j | j \in Z\}$ (Z-the set of integers) and $R^M = \{(a_i, a_j) | i < j\} \cup \{(b_k, b_\ell) | k < \ell\} \cup \{(a_r, b_s) | s \geq r+1\}$ and let $\Vdash_1$ be 1-finite forcing relation. Then, obviously, $\exists v \forall u\ (v = u \vee R(v,u) \vee (R(u,v)) \in F(M)$ but, what is less obvious, $\neg \exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \notin F(M)$. In fact there is no condition $p \subset D_1(M)$ which would force that sentence because $T \cup D_1(M) \cup \{\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)\}$ is consistent.

(b) Let all conditions be as in (a) with the only exception that now $R^M = \{(a_i, a_j) | i < j\} \cup \{(b_i, b_j) | i < j\}$. Then, again, $\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \in F(M)$ but this time also $\neg \exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \in F(M)$. For $\{\phi\} \Vdash_1 \neg \exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v))$ where $\phi \in D_1(M)$ is a (Π_1-) sentence which "claims" that there is no element which is simultaneously in one of the relations $=$, R, R^{-1} (not necessarily the same) with both a_o and b_o.

COROLLARY 1.6. *Let* M *be a model of* $T \cap \Pi_{n+1}$, $<A,f>$ *an assignment of constants to* M *and* $\phi(v_o, \ldots, v_m)$ *a* Σ_{n+1} *formula. Then for all closed terms* $t_o, \ldots, t_m$ *(of the language* $L(A)$ *it holds that:*

(a) $M \models \phi(t_o, \ldots, t_m)$ *implies: for some condition* $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi(t_o, \ldots, t_m)$;

(b) *If for some condition* $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi(t_o, \ldots, t_m)$ *then there exists an n-elementary extension* N *of* M *such that* $N \models T \cup \{\phi(t_o, \ldots, t_m)\}$

P r o o f: (a) is a consequence of theorems 0.1 and 1.3 and the obvious fact: if $p \Vdash_n \neg\neg\psi(t)$ for some closed term t ($\psi(v)$ is an arbitrary formula) then also $p \Vdash_n \neg\neg \exists v \psi(v)$ and

(b) is a consequence of Corollary 1.2.

The next example partially corrects result 3.4 from [2] (we have just given the right version). Namely, from $p \Vdash_n \neg\neg\phi$, where $p \subset D_n(M)$ and ϕ is a Σ_{n+1} sentence, $M \models \phi$ does not necessarily follow. At the same time this cor-

rection leaves in effect the part of the proof of 3.24 (iii) (from [2] in which 3.4 is used (clearly we also have at our disposal other possibilities to prove that assertion).

EXAMPLE 1.7. Once again T is the theory of (strict) order. For its model we take $\langle Z,<\rangle$ while $\Vdash_1$ is 1-finite forcing relation. Now $Z \not\models \exists v \forall u\ (u = v \vee R(u,v))$ though $p \equiv \{\forall u \forall v\ (u = v \vee R(u,v) \vee R(v,u))\} \Vdash_1 \neg\neg\exists v \forall u\ (u = v \vee R(u,v))$. The latter claim holds because if $q \supseteq p$ and $M \models T \cup q$ then M is linearly ordered whence either it itself has the last element or it is existentially complete in some model with the last element.

§ 2. Let, for a given $n > 0$, $\Phi'_n(A)$, $\Phi''_n(A)$ and C'_n, C''_n be, respectively, sets $\{\phi \in \Phi_n(A) \mid sub(\phi) \subset \Phi_n(A)\}$, $\{\phi \in \Phi_n(A) \mid \phi$ is in prenex normal form$\}$, that is $\{p' \subset SENT(\Phi'_n(A)) \mid p'$ is a finite subset consistent with $T\}$, $\{p'' \subset SENT(\Phi''_n(A)) \mid p''$ is a finite set consistent with $T\}$. Further, let $\Vdash'_n \subseteq C'_n \times SENT(L(A))$, $\Vdash''_n \subseteq C''_n \times SENT(L(A))$ be forcing relations defined in the same way as n-finite forcing relation (consequently, the sets C'_n, C''_n are ordered by inclusion and for $\phi \in AT(L(A))$ and $p \in C'_n(C''_n)$ $p \Vdash'_n \phi$ ($p \Vdash''_n \phi$) iff $\phi \in p$).

LEMMA 2.1. *For all conditions* $p = \{\phi_1,\ldots,\phi_k\} \in C_n$, $p' = \{\phi'_1,\ldots,\phi'_k\} \in C'_n$ *and* $p'' = \{\phi''_1,\ldots,\phi''_k\} \in C''_n$ *such that* ϕ_i, ϕ'_i *and* ϕ''_i, $i = 1,\ldots,k$ *are equivalent with regard to* T, *and for each sentence* $\phi \in SENT(L(A))$ *it holds that:*

$$p \Vdash_n \neg\neg\phi \quad iff \quad p' \Vdash'_n \neg\neg\phi \quad iff \quad p'' \Vdash''_n \neg\neg\phi.$$

P r o o f: By induction on the complexity of ϕ. We shall only prove, just for an illustration

$$p \Vdash_n \neg\neg\phi \quad \text{iff} \quad p' \Vdash'_n \neg\neg\phi.$$

If ϕ is an atomic formula, we can check directly

$$p \Vdash_n \neg\neg\phi \ \text{ iff } \ T \cup p \vdash \phi \ \text{ iff } \ T \cup p' \vdash \phi \ \text{ iff } \ p' \Vdash'_n \neg\neg\phi.$$

The case ϕ is $\phi_1 \wedge \phi_2$ is trivial.

If ϕ is $\neg\phi_1$, $p \Vdash_n \neg\phi_1$ and $q' \supseteq p'$, $q' \in C'_n$ then q' cannot force $\phi_1 (q' \not\Vdash'_n \phi_1)$ (for on the contrary, by inductive assumption, $q = p \cup (q'-p') \Vdash_n \neg\neg\phi_1$) whence $p' \Vdash'_n \neg\phi_1$. Also, from $p' \Vdash'_n \neg\phi_1$ follows $p \Vdash_n \neg\phi_1$.

Let there now be $\phi \equiv \exists v\phi_1(v)$, $p \Vdash_n \neg\neg\exists v\phi_1(v)$ and $q' \supseteq p'$. For $q = p \cup (q'-p')$ there exists a condition $r \in C_n$, $r \supseteq q$ and a closed term t such that $r \Vdash_n \phi_1(t)$. By the inductive hypothesis $r' = q' \cup (r-q)' \Vdash'_n \neg\neg\phi_1(t)$. Thus $p' \Vdash'_n \neg\neg\exists v\phi_1(v)$. Analogously one would prove the opposite.

COROLLARY 2.2. $T^{f_n} = T^{f'_n} = T^{f''_n}$ *where* $T^{f_n}, T^{f'_n}$ *and* $T^{f''_n}$ *are forcing companions (of* T*) corresponding to, respectively, the forcing relations* $\Vdash_n$, $\Vdash'_n$ *and* $\Vdash''_n$.

Let us remark that with 2.1 we have proved also the following result: if $p = \{\phi_1,\dots,\phi_k\}$ and $q = \{\theta_1,\dots,\theta_k\}$ are elements of the set $C_n (C'_n, C''_n)$ and if $T \vdash \phi_i \leftrightarrow \theta_i$, $i = 1,\dots,k$ then for any sentence ψ of the language $L(A)$ $p \Vdash_n \neg\neg\psi$ iff $q \Vdash_n \neg\neg\psi$ ($p \Vdash'_n \neg\neg\psi$ iff $q \Vdash'_n \neg\neg\psi_1$, $p \Vdash''_n \neg\neg\psi$ iff $q \Vdash''_n \neg\neg\psi$).

For given forcing relations there also holds (apparently stronger) the proposition (because of 2.1 it is sufficient to give the proof for forcing relation $\Vdash_n$):

LEMMA 2.3. *Let* p *and* q *be conditions (from* C_n*) and* $T \vdash \bigwedge p \leftrightarrow \bigwedge q$. *Then for each* $\phi \in SENT(L(A))$ $p \Vdash_n \neg\neg\phi$ *iff* $q \Vdash_n \neg\neg\phi$. *In particular* $T^{f_n}[p] = T^{f_n}[q]$.

P r o o f. Let us suppose $p(c_o,\dots,c_m) \Vdash_n \neg\neg\phi(c_o,\dots,c_m)$ where $c_o,\dots,c_m$ are all constants from A which occur in either the sentences of p or in ϕ. By the known theorem (the proof of which, among other thing, uses corollary 1.2) $\forall v_o \dots \forall v_m(\bigwedge p(v_o,\dots,v_m) \to \phi(v_o,\dots,v_m)) \in T^{f_n} \subseteq T^{f_n}[q]$ and so $q \Vdash_n \neg\neg(\bigwedge p(c_o,\dots,c_m) \to \phi(c_o,\dots,c_m)$. Since for

each $\psi \in p$ $T \cup q \vdash \psi$ according to 2.5 $q \Vdash_n \neg\neg\psi$, whence $q \Vdash_n \bigwedge\neg\neg p$ and consequently $q \Vdash_n \neg\neg\bigwedge p$. It follows that $q \Vdash_n \neg\neg\phi$.

Still we are obliged to prove theorem 1.1 and corolary 1.2. But the analogies of these propositions hold for the forcing relations $\Vdash'_n$ (and set $\Phi'_n(A)$) (Theorem 0.2), hence the proofs of 1.1 and 1.2 follow from 0.1 and 2.1. Let us demostrate it for the case: (for $p \in C_n$ and $\phi \in \Phi_n(A)$) $\phi \in p$ implies $p \Vdash_n \neg\neg\phi$.

Let $p = \{\phi_1,\ldots,\phi_k\}$ and $\phi \equiv \phi_i$ for some i, $1 \le i \le k$ and let $p' = \{\phi'_1,\ldots,\phi'_k\} \in C'_n$ where $\vdash \phi_j \leftrightarrow \phi'_j$ $j = 1,\ldots,k$. By 0.2 $p' \Vdash'_n \neg\neg\phi'_i$ and by 0.1 $p' \Vdash'_n \neg\neg(\phi'_i \to \phi_i)$. Thus $p' \Vdash'_n \neg\neg\phi_i$ and then also (2.1) $p \Vdash \neg\neg\phi_i$.

Clearly the assertions corresponding to 1.1 and 1.2 hold for forcing relation $\Vdash''_n$ (and set $\Phi''_n(A)$) as well.

Let now C, C_o, C' and C'_o be, in order, the sets (of conditions)

$\{p|p$ is a finite set of the basic sentences of the language $L(A)$, consistent with $T\}$

$\{p|p$ is a finite subset of $SENT(\Phi_o(A))$, consistent with $T\}$,

$\{p|p$ is a finite set consistent with T, elements of which are conjunction of basic sentences of the language $L(A)\}$ and $\{p|p$ is a finite set of quantifier free sentences of the language $L(A)$ in disjunctive normal form, consistent with $T\}$

and let $\Vdash$, $\Vdash_o$, $\Vdash'$ and $\Vdash'_o$ and T^f, T^{f_o}, $T^{f'}$ and $T^{f'_o}$ be, respectively, corresponding to these sets, forcing relations i.e. forcing companions (in any case we recall: the sets are ordered by inclusion and in all cases p forces an atomic sentence ϕ if (and only if) ϕ belongs to p); $\Vdash$ is Robinson's finite forcing relation and $\Vdash_o$ is O-finite forcing relation.

By Lemma 2.1 it follows immediately that:

LEMMA 2.4. $T^{f_o} = T^{f'_o}$.

LEMMA 2.5. *For* $p' \in C'$ *let* $p(\in C)$ *be the set of all basic sentences which occur as subformulas of sentences of* p'. *Then it holds that for each* $p' \in C'$ *and each* $\phi \in SENT(L(A))$

$$p' \Vdash' \neg\neg\phi \text{ if and only if } p \Vdash \neg\neg\phi.$$

P r o o f. By induction on the complexity of formula ϕ.

COROLLARY 2.6. $T^f = T^{f'}$.

LEMMA 2.7. *If* $p' = \{\vee\Phi_{1_i}, \dots, \vee\Phi_{k_i}\} \in C'_o$ $(\Phi_{m_i}$ *is the conjunction of basic sentences) and* $p \in C' \subseteq C'_o$ *is a condition, elements of which are arbitrarily chosen "representative" disjuncts from each sentence of* p' *(it can happen that some "representatives" coincide) then for any sentence* $\phi \in SENT(L(A))$

$$p' \Vdash'_o \neg\neg\phi \text{ implies } p \Vdash'_o \neg\neg\phi.$$

P r o o f. Obviously, for if $p \subseteq q' \in C'_o$ then $q' \cup p'$ is a condition, too.

COROLLARY 2.8. *If* $p \in C' \subseteq C'_o$ *and* $\phi \in SENT(L(A))$ *then* $p \Vdash' \neg\neg\phi$ *if and only if* $p \Vdash'_o \neg\neg\phi$.

COROLLARY 2.9. $T^{f'} = T^{f'_o}$.

THEOREM 2.10. $T^f = T^{f_o}$.

Therefore if we look at the (finite) forcing relation, before all, as a tool for obtaining the forcing companion and other, for this theory, relevant results, we are free in the choice of any of the four given sets (C, C_o, C', C'_o) for the set of conditions, that is when an n-finite forcing relation is in question, $n > 0$, we can choose between C_n, C'_n and C''_n (of course in both cases we could enlarge the assortment of the sets of conditions).

§ 3. For obtaining n-finite forcing companion $(n > 0)$ we do not need more than a finite forcing relation. Moreover each theory T of a language L is contained in (some) theory defined in (an appropriate) expansion of L for which the finite forcing and n-finite forcing companions coincide. There follow the proofs of these assertions.

Let us join to each formula $\phi(v_{i_1},\dots,v_{i_m})$ from Φ_n, where $fv(\phi) = \{v_{i_1},\dots,v_{i_m}\}$ and $(v_{i_1},\dots,v_{i_m})$ is uniquely determined (for instance, by a sequence of free occurences of variables v_{i_k}, $i \le k \le m$ in ϕ), a new relation symbol $R_{\phi,\tilde{v}}$ of length m ($R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ is then always interpreted as a result of substituting in $R_{\phi,\tilde{v}}(v_{i_1},\dots,v_{i_m})$ the terms $t_1,\dots,t_m$ for occurences of $v_{i_1},\dots,v_{i_m}$, respectively); in case ϕ is a sentence its corresponding relation symbol (now in notation just R_ϕ) is of length one. In the language L' obtained by extension of the language L by the set of these new relation symbols let T' be the following set of sentences:

$$T \cup \{\forall v_{i_1},\dots,\forall v_{i_m}(\phi(v_{i_1},\dots,v_{i_m}) \leftrightarrow R_{\phi,\tilde{v}}(v_{i_1},\dots,v_{i_m}) \mid$$

$$\mid \phi \in \Phi_n - SENT(\Phi_n)\} \cup \{(\phi \leftrightarrow \forall v_o R_\phi(v_o)) \wedge (\forall v_o R_\phi(v_o) \vee \forall v_o \neg R_\phi(v_o)) \mid$$

$$\mid \phi \in SENT(\Phi_n)\}.$$

LEMMA 3.1. T' *is consistent.*

P r o o f. Clearly. Any model M of T can be extended to a model M' (with the same domain) for T' by interpreting the new relation symbols $R_{\phi,\tilde{v}}$ by $R^{M'}_{\phi,\tilde{v}}$ where $(a_1,\dots,a_m) \in R^{M'}_{\phi,\tilde{v}}$ if and only if $M \models \phi[a_1,\dots,a_m]$, while $R^{M'}_\phi = M$ if $M \models \phi$, otherwise $R^{M'}_\phi = \emptyset$.

We immediately notice the following

$$T' \vdash \forall v_o R_\phi(v_o) \leftrightarrow \exists v_o R_\phi(v_o)$$

hence also

$$T' \vdash \forall v_o R_\phi(v_o) \leftrightarrow R_\phi(t) \quad \text{for any closed term } t.$$

Let us note that for any basic sentence of the language $L'(A)$ there exists an atomic sentence of the form $R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ or $R_\phi(t)$ equivalent to it with regard to T' (this is implied by the fact that the set Φ_n is closed under negation).

Let C' be the set of conditions of Robinson's finite forcing relation ($\Vdash'$) for the theory T' and language $L'(A)$ and let $C'' = \{p' \in C' \mid$ the elements of p' are sentences of the form $R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ or $R_\phi(c)$ where $t_1,\dots,t_m$ are (closed) terms of $L'(A)$ and c is a fixed, but otherwise, arbitrarily chosen constant from $L(A)$ $\Vdash_n$ and C_n will remain the signs of n-finite forcing and its set of conditions for the theory T and language $L(A)$. For $p' = \{R_{\phi_1,\tilde{v}^1}(\tilde{t}^1),\dots,$ $R_{\phi_m,\tilde{v}^m}(\tilde{t}^m), R_{\psi_1}(c),\dots,R_{\psi_k}(c)\} \in C''$ $(m \geq 0,\ k \geq 0)$ let $f(p') = \{\phi_1(\tilde{t}^1),\dots,\phi_m(\tilde{t}^m),\psi_1,\dots,\psi_k\} (\in C_n)$. Obviously f is a surjective mapping of C'' onto C_n (in case L is a language without constants f is injective as well). $q' \in f^{-1}(p)$ means that $f(q') = p$. Clearly for $p', q' \in f^{-1}(p)$ $T' \vdash \bigwedge p' \leftrightarrow \bigwedge q'$.

THEOREM 3.2. *For each* $\phi \in SENT(L(A))$ *and each* $p' \in C''$ *it holds that*

$$f(p') \Vdash_n \neg\neg\phi \quad \textit{if and only if} \quad p' \Vdash' \neg\neg\phi.$$

P r o o f. By induction, on the complexity of ϕ.

If ϕ is an atomic sentence then $T \cup f(p') \vdash \phi$ iff $T' \cup p' \vdash \phi$, hence also $f(p') \Vdash_n \neg\neg\phi$ iff $p' \Vdash' \neg\neg\phi$.

The case ϕ is $\phi_1 \wedge \phi_2$ is trivial.

Let us suppose now that $\phi \equiv \neg\phi_1$ and $f(p') \Vdash_n \neg\phi_1$ but $p' \nVdash' \neg\phi_1$. According to the already concluded facts (and Lemma 2.3) there exists $q' \in C''$ such that $q' \supseteq p'$ and $q' \Vdash' \neg\neg\phi_1$. It follows by the inductive assumption $f(q') \Vdash' \neg\neg\phi_1$, which

is, however, in contradiction with $f(p') \Vdash_n \neg\phi_1$. On the other hand, if $p' \Vdash' \neg\phi_1$ but not $f(p') \Vdash_n \neg\phi_1$ then for some $q \in C_n$, $q \supseteq f(p')$ $q \Vdash_n \neg\phi_1$. Let $q' \in C'$ be such a condition that $q' \in f^{-1}(q)$ and $p' \subseteq q'$. Then (again we keep in mind 2.3 and the above remarks) $q' \Vdash' \neg\neg\phi_1$ and there is a contradiction.

The proof for case $\phi \equiv \exists v\phi_1(v)$ is only technically somewhat more complicated. Let us suppose $f(p') \Vdash_n \neg\neg\exists v\phi_1(v)$ and let $p' \subseteq q'' \in C'$. Let us obtain a condition $(p' \subseteq) q' \in C''$ by substituting the basic sentences of $L(A)$ and negations of the atomic sentences of $L'(A)$ in q'' by the equivalent to them (with regard to T') atomic sentences of the form $R_{\phi,\tilde{v}}(\tilde{t})$, that is, $R_\psi(c)$ and by substituting c for t in sentences of the form $R_\theta(t)$. If $r \in C_n$ and (a closed term) t are such that $f(q') \subseteq r$ and $r \Vdash_n \phi(r)$ and r' is a condition such that $q' \subseteq r' \in f^{-1}(r)$, then $r' \Vdash' \neg\neg\phi_1(t)$ whence $r'' = q'' \cup (r'-q') \Vdash' \neg\neg\phi_1(t)$ and so $p' \Vdash' \neg\neg\exists v\phi_1(v)$. Finally, let $p' \Vdash' \neg\neg\exists v\phi_1(v)$ and $q \supseteq f(p')$. If $p' \subseteq q' \in f^{-1}(q)$ there exists a condition $r'' \in C'$ and a closed term t sich that $q' \subseteq r''$ and $r'' \Vdash' \phi_1(t)$. Then $(q' \subseteq) r' \Vdash' \neg\neg\phi_1(t)$ where the condition $r' \in C'$ is obtained (from r'') in the same way as q' (from q'') in the previous case and thus by the inductive hypothesis as well $(q \subseteq) f(r') \Vdash_n \neg\neg\phi_1(t)$. Consequently, $f(p') \Vdash_n \neg\neg\exists v\phi_1(v)$.

COROLLARY 3.3. $T^{f_n} = (T')^f \cap SENT(L)$.

Let us define, recursively (and simultaneously), the sequences of languages L^k and theories T^k, $k \in \omega$ in the following way:

$$L^0 = L, \quad T^0 = T$$

$L^{k+1} = (L^k)'$, $T^{k+1} = (T^k)'$ (where it is assumed that $(L^k)'$ and $(T^k)'$ are being formed by extension of the language L^k, that is, theory T^k, in the way analogous to obtaining L' and T' (in the previous proposition) from L and T).

Let $L^{\omega} = \bigcup_{k\in\omega} L^{k}$ and $T^{\omega} = \bigcup_{k\in\omega} T^{k}$.

LEMMA 3.4. *T^{ω} is consistent.*

LEMMA 3.5. *Let C^{ω} and C^{k}, $k \in \omega$, be, respectively, the sets of conditions of Robinson's finite forcing relations for theory T^{ω} and language $L^{\omega}(A)$, that is, for theory T^{k} and language $L^{k}(A)$. Then $C^{\omega} = \bigcup_{k\in\omega} C^{k}$*

P r o o f. Clearly. Any model of theory T^{k} can be expanded to a model of theory T^{ω}.

Surely the following holds as well

LEMMA 3.6. *Let C_n^{ω} and C_n^{k}, $k \in \omega$ be, one after another, the sets of conditions of n-finite forcing relations for theory T^{ω} and language $L^{\omega}(A)$, that is, for theory T^{k} and language $L^{k}(A)$. Then $C_n^{\omega} = \bigcup_{k\in\omega} C_n^{k}$.*

LEMMA 3.7. *For each $\phi \in \mathrm{SENT}(L^{\omega}(A))$ and each $p \in C^{\omega}$*

$$p \Vdash \neg\neg\phi \text{ if and only if } p \Vdash_n \neg\neg\phi$$

where $\Vdash$ and $\Vdash_n$ are finite, that is, n-finite forcing relation for theory T^{ω} and language $L^{\omega}(A)$.

P r o o f. By induction on the complexity of formula ϕ.

The cases: ϕ is an atomic sentence and $\phi \equiv \phi_1 \wedge \phi_2$ are trivial. The case where ϕ is $\neg\phi_1$ is slightly more difficult, and the case where ϕ is $\exists v\phi_1(v)$ demands (as well as in 3.2) a litlle more patience, which we shall show again.

Let $p \Vdash \neg\neg\exists v\phi_1(v)$ and $p \subseteq q \in C_n^{\omega}$, $q = p \cup \{\psi_1(\tilde{t}^1), \ldots, \psi_k(\tilde{t}^k)\}$, $k \geq 0$. Further let $\overline{\psi}_1(\tilde{t}^1), \ldots, \overline{\psi}_k(\tilde{t}^k)$ be atomic sentences such that $T^{\omega} \vdash \psi_i(\tilde{t}^i) \leftrightarrow \overline{\psi}_i(\tilde{t}^i)$, $i = 1, \ldots, k$ (obviously we have at our disposal a wide choice of such sentences, but any choice is equally good for this proof), and let $r \in C^{\omega}$ be a condition and t a closed term such that

$p \cup \{\overline{\psi}_1(\tilde{t}^1),\dots,\overline{\psi}_k(\tilde{t}^k)\} \subset r$ and $r \Vdash \phi_1(t)$. By the inductive hypothesis $r \Vdash_n \neg\neg\phi_1(t)$, whence, according to 2.3, $q \cup r - \{\overline{\psi}_1(\tilde{t}^1),\dots,\overline{\psi}_k(\tilde{t}^k)\} \Vdash_n \neg\neg\phi_1(t)$. Accordingly $p \Vdash_n \neg\neg\exists v\phi_1(v)$. But if $p \Vdash_n \neg\neg\exists v\phi_1(v)$ and $p \subseteq q \in C^\omega$ then for some $r \in C^\omega_n$ and some closed term t, $q \subseteq r$ and $r \Vdash_n \phi_1(t)$. Let r' be a condition made by substituting sentences from $r-q$ by atomic sentences, which are equivalent to them with regard to T^ω. Then $r' \Vdash_n \neg\neg\phi_1(t)$ and also (by the inductive assumption) $r' \Vdash \neg\neg\phi_1(t)$. We conclude $p \Vdash \neg\neg\exists v\phi_1(v)$.

THEOREM 3.8. $(T^\omega)^f = (T^\omega)^{f_n}$

P r o o f. An immediate consequence of Lemmas 2.3 and 3.7.

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REZIME

O n-KONAČNOM FORSINGU

U [2] smo manje-više samo konstatovali da se (osnovna) svojstva Robinsonovog forsinga prirodno prenose na n-konačni forsing (u smislu njihovog "transliranja za n") (oko dokaza se sasvim prirodno nismo mnogo trudili jer su do poslednjeg inspirisani odgovarajućim za konačni forsing). Kako smo, medjutim, tom prilikom, bez posebnog objašnjenja koristili i (jedan) rezultat iz [4] (1.5 u [2]) (opšteg karaktera) koji, pak, nije direktno primenljiv, s obzirom da smo za uslove uzimali podskupove skupa svih rečenica koje su ekvivalentne rečenicama u preneks normalnoj formi sa najviše n blokova kvantifikatora (takav skup (C_n) ne ispunjava: $\phi \in C_n \rightarrow sub(\phi) \subset C_n$) ovde ćemo se time podrobnije pozabaviti. Ukazaćemo ujedno na deo "asortimana" čiji elementi nam se (ravnopravno) nude za skup uslova kao i moguću simplifikaciju i poboljšanje pojedinih dokaza i stavova, ali i delom korigovati dva rezultata iz [2].

Sam n-konačni forsing za $n > 0$, gledano sa čisto "tehničkog" aspekta ne nudi nam mnogo novog, naime n-konačno forsing pridruženje (date teorije) se može dobiti i primenom Robinsonovog konačnog forsinga. Posledica toga je i da za svaku teoriju T definisanu u jeziku L postoji proširenje T' definisano u adekvatnom proširenju jezika $L(L')$ za koju se konačno i n-konačno forsing pridruženje podudaraju. No mišljenja smo, izučavanje njegovih svojstava i posebno svojstava korespodentnog forsing pridruženja korisno je iz više razloga (i iz gotovo svih ostalih aspekata).

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OLGA HADŽIĆ

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